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An Adaptive Finite Element Formulation for the Solution of Second Order Obstacle Problems using Quadratic Lagrange Polynomials

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Abstract: A weighted-residual based *a posteriori* error estimation formulation in Galerkin's finite element fashion using quadratic Lagrange polynomials has been formulated to find numerical solutions of obstacle, unilateral and contact second-order boundary-value problems. The approach having piecewise quadratic shape functions has been utilized for checking the approximate solutions for spatially adaptive finite element grids. The local element balance based on the residual has been considered as an error assessment criterion. Numerical testing indicates that local errors are large at the interface regions where the gradients are large. A comparison of an adaptive refined grid with that of a uniform mesh for second order obstacle boundary value problems, confirms the superiority of the adaptive scheme without increasing the number of unknown coefficients.

Keywords: Adaptive grid refinement scheme, Quadratic Lagrange polynomials, Galerkin's method, Finite element method, Boundary-value problems

Mathematics Subject Classification: 65M60

1 Introduction

Appropriate grid construction is a key characteristic of any formulation intended for solving differential equations using finite element methods (FEM). It is observed in most situations that, once a grid has been generated, it is kept fixed all the way through the enduring computations. This may cause an increase in local errors in the solution domain where the element size "h" is not sufficient

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to model a rapidly changing solution correctly. The adaptive grid generation technique helps in creating a large number of grid nodes in the regions where the solution is sharply changing; it can lessen the local errors and also a small grid density in domains where the solution is either constant or slowly varying, can facilitate a more efficient use of resources.

In almost all adaptive grid generation schemes, the local error assessment criterion is the key concern. Simple post processing error estimators have been used by Zienkiewicz and Zhu [1, 2] based on the approaches proposed by Oden [3] and Hinton [4]. An adaptive scheme based on *a posteriori* error estimation was employed by Mirza et al [5], has also been employed in our present investigations for the obstacle problem considered. Since this scheme depends on the governing differential equations of the specific physical phenomenon, therefore, it is more precise and vigorous than the other schemes presented by Babuska et al [6, 7], and Carey [8].

In this article, an adaptive technique based on *a posteriori* error estimation has been utilized to solve boundary value problems using weighted residual formulations. Quadratic shape functions are used for obtaining smooth approximations to the solution of a system of second-order boundary-value problems [9, 10, 11, 12, 13] of the following type:

$$y'' = \begin{cases} f(x) & a \leq x < c, \\ g(x)y(x) + f(x) + r, & c \leq x < d, \\ f(x), & d \leq x \leq b. \end{cases} \tag{1}$$

with boundary conditions,

$$y(a) = \alpha_1 \quad \text{and} \quad y(b) = \alpha_2 \tag{2}$$

where $f(x)$ is a given force on the string and the continuity conditions of y and y' at c and d. Here, f and g are continuous functions on $[a, b]$ and $[c, d]$ respectively. The parameters $\alpha_1, \alpha_2,$ and r, are real finite constants. Such types of systems arise in the study of obstacle, unilateral, moving and free boundary value problems [9, 10, 11, 12, 13] describing the equilibrium relationship of an obstacle string pulled at the ends and lying over an elastic step. The following system of differential equations is obtained for the same equilibrium configuration of an obstacle string pulled at the ends and lying over an elastic step:

$$y'' = \begin{cases} f(x) & 0 \leq x < \pi/4 \text{ and } 3\pi/4 \leq x \leq \pi, \\ y + f(x) - 1, & \pi/4 \leq x < 3\pi/4, \end{cases} \tag{3}$$

with boundary conditions,

$$y(0) = y(\pi) = 0, \tag{4}$$

Here x is the independent variable, $y(x)$ is the unknown function (state variable). We will call $f(x)$ an interior load (since it represents load applied to the interior (*i.e.*, the domain)) of the system. The domain is any finite or infinite interval along the x-axis. The word 'load' is to mean any agent or driving force that causes the state of the system to change, for example, a force, displacement, voltage or temperature etc. In general it is not possible to obtain the analytical solution of (1) for arbitrary choices of $f(x)$ and $g(x)$, thus, usually numerical methods are employed for obtaining an approximate solution of (1).

One of the main advantages of the adaptive technique is the attainment of higher accuracy. Once the solution has been computed, the information required for FEM interpolation between mesh points is available. This is particularly important when the solution of the boundary-value problem is required at different locations in the interval $[a, b]$.

The layout of the paper follows with section 2, where an adaptive finite element formulation is presented using quadratic Lagrange polynomial. Computational aspects are given in section 3 and the comparison of the uniform grid solutions and adaptive grid solutions are discussed in section 4.

2 An Adaptive Finite Element Formulation

The finite element method provides an elegant and systematic technique for constructing basis functions for Galerkin's approximations of boundary value problems; a brief reflection reveals that the idea also provides a basis for methods of interpolation. The method leads to the Lagrange families of finite elements.
in this formulation C^0-quadratic elements are used and require three nodes to uniquely define a quadratic polynomial. One node must be located on the element boundary, in order to simplify assembly and to make sure that the resulting assembled trial functions are local. The third node may be located anywhere in the interior. The middle node plays no role in establishing inter-element continuity; its only purpose is to help in defining a quadratic polynomial. We prefer to recast the quadratic trial function in terms of values of the dependent functions at nodes i, j&k (the convention used by [5, 14-17]) thus

$$\tilde{y}(x) = N_1 y_i + N_2 y_j + N_3 y_k \equiv [N]\{\tilde{y}\} \tag{5}$$

Here $\{\tilde{y}\}^T = \begin{bmatrix} y_i & y_j & y_k \end{bmatrix}$ is the vector of nodal coordinates and $[N] = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix}$ is called the vector of interpolation, shape, or basis functions, where $N_1 = (x - x_j)(x - x_k) \ / \ (x_i - x_j)(x_i - x_k)$, $N_2 = (x - x_i)(x - x_k) \ / \ (x_j - x_i)(x_j - x_k)$, and $N_3 = (x - x_i)(x - x_j) \ / \ (x_k - x_i)(x_k - x_j)$. Now, the governing differential equation is of the form (see equation (3)):

$$y'' - \alpha y - f(x) + \beta = 0 \tag{6}$$

where

$$\begin{array}{ll} \alpha = \beta = 0, & \text{for } \ 0 \le x < \pi/4 \ \text{ and } \ 3\pi/4 \le x \le \pi, \\ \alpha = \beta = 1, & \text{for } \ \pi/4 \le x < 3\pi/4, \end{array} \tag{7}$$

Galerkin's finite element formulation as given in [14, 15, 16, 17], is used for our particular problem and after substituting the trial functions, the equation (6) can be written in discretized form as

$$w\tilde{y}'|_{X_1}^{X_2} - \sum_{e=1}^{n} \left(\int_{X_e} w'\tilde{y}'dx + \alpha \int_{X_e} w\tilde{y}dx + \int_{X_e} wf(x)dx - \beta \int_{X_e} wdx \right) = 0 \tag{8}$$

where $'e'$ represents the element and $'n'$ represents the total number of elements in the discretized region. Now, when the element equations are assembled according to equation (8). The equations for the elements must combine in such a manner that only the boundary terms for the element nodes on the boundary will contribute; all other terms for the interior nodes will be zero. This implies that the boundary terms for the elements at common interior nodes cancel each other. In this particular problem (3), the contribution of the boundary terms will also vanish due to $y(0) = 0$ and $y(\pi) = 0$. Also, $f(x) = 0$ for the given system of second-order differential equations in the example. Therefore, in matrix notation the system of equations (8) can be written as

$$\sum_{e=1}^{n} (k_1\tilde{y} + \alpha k_2 \tilde{y} - \beta f_1) = 0 \tag{9}$$

where $k_1 = \frac{1}{3L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$, $k_2 = \frac{L}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}$, and $f_1 = \frac{L}{6}\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$. Note that k_1 and k_2 are stiffness matrices, whilst f_1 is a force vector. $L = x_k - x_j$ or $L = x_j - x_i$ is the length between two adjacent nodes. Using the conditions given in (7), we can write (9) for one element as

$$\begin{array}{ll} k_1\tilde{y} = 0, & \text{for } \ 0 \le x < \pi/4 \ \text{ and } \ 3\pi/4 \le x \le \pi, \\ k_1\tilde{y} + \alpha k_2\tilde{y} - \beta f_1 = 0, & \text{for } \ \pi/4 \le x < 3\pi/4, \end{array} \tag{10}$$

Then applying the assembly procedure given in [14-17] for $'n'$ elements and using the conditions of equation (10), we will get global system of equations

$$K_1\tilde{y} = F_1 \tag{11}$$

where K_1 is a global stiffness matrix and F_1 is global force vector.

2.1 Residual Based *a posteriori* Error Estimation for Adaptive Grid Refinement Approaches

The adaptive grid refinement stratagem progressively refines the grid in appropriate regions of the solution domain. The vital part of such an approach is an error assessment criterion for checking the quality of the approximate solution for various finite element grids. An *a posteriori* error assessment criterion, which generates error estimates during the course of finding an approximate solution and adaptively changes the grid is used [1-5].

2.2 A Residual Based *a posteriori* Scheme

A local error evaluation system which is based on the estimation of local residuals of the differential equation is presented. A local element balance equation can be obtained from equation (8) of the obstacle problem over the length of an element, *i.e.*,

$$\int_{X_1^e}^{X_2^e} w^{/}\tilde{y}^{/}dx + \alpha\int_{X_1^e}^{X_2^e} w\tilde{y}dx + \int_{X_1^e}^{X_2^e} wf(x)dx - \beta\int_{X_1^e}^{X_2^e} wdx \approx 0 = \Re^e \tag{12}$$

A global particle balance over the whole system demands the condition

$$\sum_{e=1}^{n}\left(\int_{X_e} w^{/}\tilde{y}^{/}dx + \alpha\int_{X_e} w\tilde{y}dx + \int_{X_e} wf(x)dx - \beta\int_{X_e} wdx\right) \approx 0 = \Re, \tag{13}$$

where the summation is over all the elements of the system.
In obstacle problems, the conditions are given by (12) and (13) and can be used as assessment criterion for checking the closeness of the approximate solution to the exact solution. In case of an approximate solution, these conditions lead to a non-zero residual. However, the element-wise condition can be employed as an assessment criterion for dividing elements into elements of smaller size. In general, for an approximate solution

$$k_1\tilde{y} + \alpha k_2\tilde{y} - \beta f_1 \approx 0 = \Re. \tag{14}$$

Using this local residual as an assessment criterion, an algorithm can be developed which locates elements where residuals are large, locally refines the grid and computes the solution on the newly generated grid. This algorithm can be repeated until a specified stopping measure is met. While dividing an element into sub-elements, the number of subdivisions can be made proportional to the relative magnitude of the local residual. This helps us in distributing the residual uniformly over the entire domain.

3 Computational Aspects and Implementation

A computer code has been developed to solve obstacle, unilateral and contact second-order boundary-value problems using an adaptive finite element scheme. The program has a core module which employs quadratic Lagrange polynomials as basis functions for finite element formulation. This core module follows the general steps of a finite element solution strategy to find an approximate solution. The core module is called by the adaptive grid generator to determine local residuals and explore the possibility of grid refinements in appropriate regions of the problem. The error estimation scheme established in the previous section has been implemented in the outer grid refining iteration module. The working of this grid refining iteration module is described in the algorithm in the section that follows.

3.1 Grid Refining Iteration Algorithm

Step 1: Read the input parameters for the obstacle problem.
Step 2: Set maximum number of iterations MaxIterations, minimum local residual ε_{local}.
Step 3: For the given grid, determine the solution at the nodes.
Step 4: Compute the residual for each of the element $\Re^e$ by using the solution obtained from step 3. Also calculate the global residual $\Re$ and standard deviation $\sigma_\Re$ of the element residuals.
Step 5: Check for the stopping criteria for the adaptive scheme chosen in step 2.
Step 6: Iterate over all the elements starting from element $e = 1$,
If $\Re^e > \sigma_\Re$ then
  Split the element into sub-elements.
  Update the nodal-coordinate and element-connectivity tables.
end if
Step 7: Repeat steps 3 to 6.
The above algorithm works repeatedly, until either a convergence is reached or the maximum number of iterations is met.

4 Numerical Results and Discussions

We consider the system of differential equation (3), when $f(x) = 0$, *i.e.,*

$$y'' = \begin{cases} 0, & \text{for } 0 \le x < \pi/4 \text{ and } 3\pi/4 \le x \le \pi, \\ y - 1, & \text{for } \pi/4 \le x < 3\pi/4, \end{cases} \tag{15}$$

with boundary conditions (4). The analytical solution for the above mentioned example is given as [9, 10, 14]

$$y(x) = \begin{cases} 4x/\gamma_1, & \text{for } 0 \le x < \pi/4, \\ 1 - 4\cosh(\pi/2 - x)/\gamma_2, & \text{for } \pi/4 \le x < 3\pi/4, \\ 4(\pi - x)/\gamma_1\,, & \text{for } 3\pi/4 \le x \le \pi, \end{cases} \tag{16}$$

where $\gamma_1 = \pi + 4\coth(\pi/4)$ and $\gamma_2 = \pi\sinh(\pi/4) + 4\cosh(\pi/4)$.
A grid refining iteration algorithm is applied on the mentioned obstacle problem. Figure 1 shows a gradual reduction in the local residuals as the solution procedure gradually adapts according to the approximation solution.
It is shown in Figures 1(a)-1(d), as the algorithm proceeds, more grid points are generated in regions where the solution is changing rapidly and the local residual errors are large. Local residual errors corresponding to the approximate solutions are shown in Figures 1(e)-1(h). The algorithm automatically identifies those elements where grid refinement is required. The adaptive procedure in turn results in the reduction of local errors from values ranging from [0, 0.4] to [0, 10.7].

After obtaining a refined grid, the core module was used to find the solution using both a uniform and refined grid. The same number of grid points is used in the uniform grid as that obtained for the adaptively refined grid. A comparison of the two solutions along with local residuals is shown in Figure 2. It can be seen that the uniform and adaptive approximate solutions are similar (see Figures 2(a) and 2(b)).

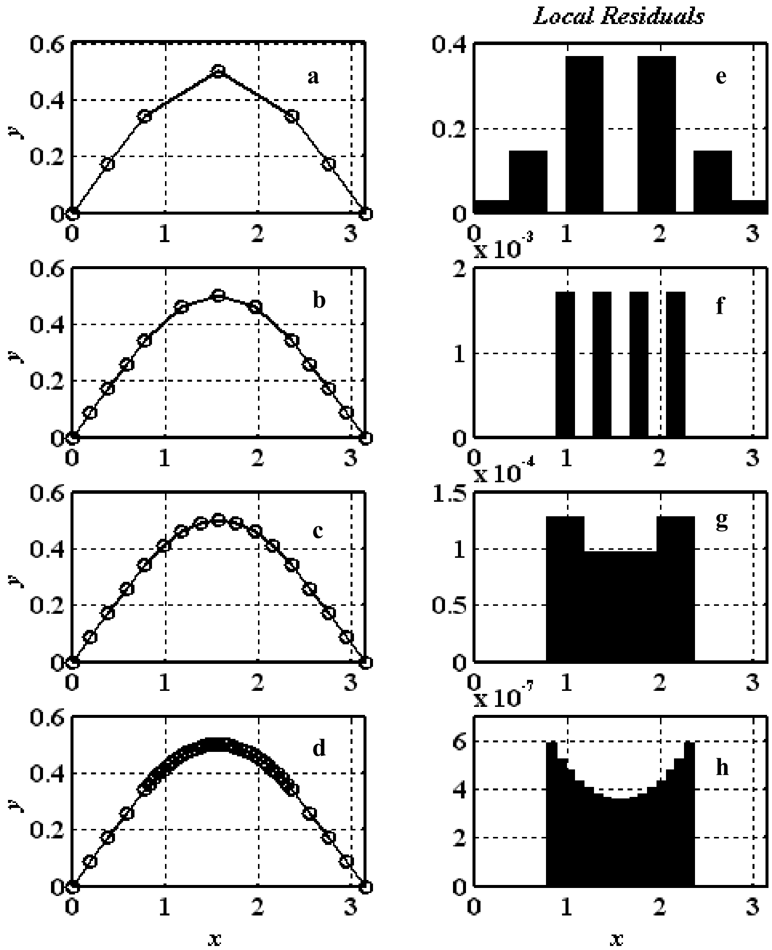

Figure 1: Adaptive FEM solutions (using quadratic Lagrange polynomials) and element wise local residuals for coarse meshing to fine meshing.

In case of a uniform grid solution, the local residual error (Figs. 2(c)) is large (in the range [0, 0.5]), while the local residual error is very small in case of the adaptively generated grid (in the range [0, 10.7]). A comparison of the two errors is also shown in Figures 3(a)-3(c) with increasing number of elements. It is shown that in the case of a uniform grid solution, the local error is large, while it is very small in the case of the adaptively generated grid.

5 Conclusions

In this paper an adaptive grid refinement approach for the finite element solution of obstacle, unilateral and contact second-order boundary-value problems has been presented. In order to examine the adaptive grid refinement approach, a computer code has been developed using *a posteriori* error assessment criteria. A noticeable reduction in the local errors is observed using the adaptive grid refinement approach which resulted in obtaining more accurate solutions in comparison with the uniform grids. The improvement in accuracy is attributed to the repositioning

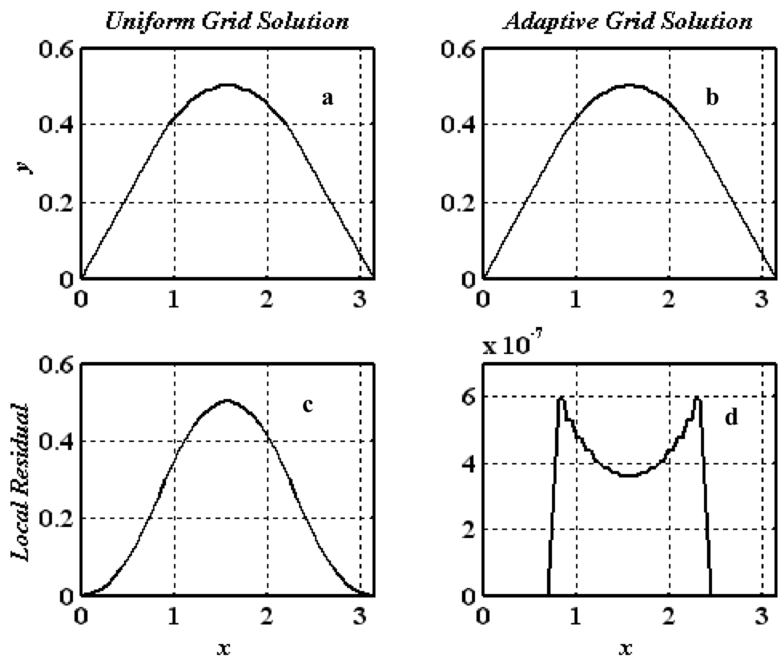

Figure 2: Uniform grid solution and adaptive grid solution and comparison of element wise local residuals (20 quadratic elements).

of the grid nodes. It is not due to simply an increase in the number of nodes. The adaptive grid refinement scheme automatically repositions the nodes in regions where the local errors are large. A comparison between the adaptive and uniform grid approaches, based on progressively increasing the number of elements indicates fast convergence of the adaptive grid refinement technique.

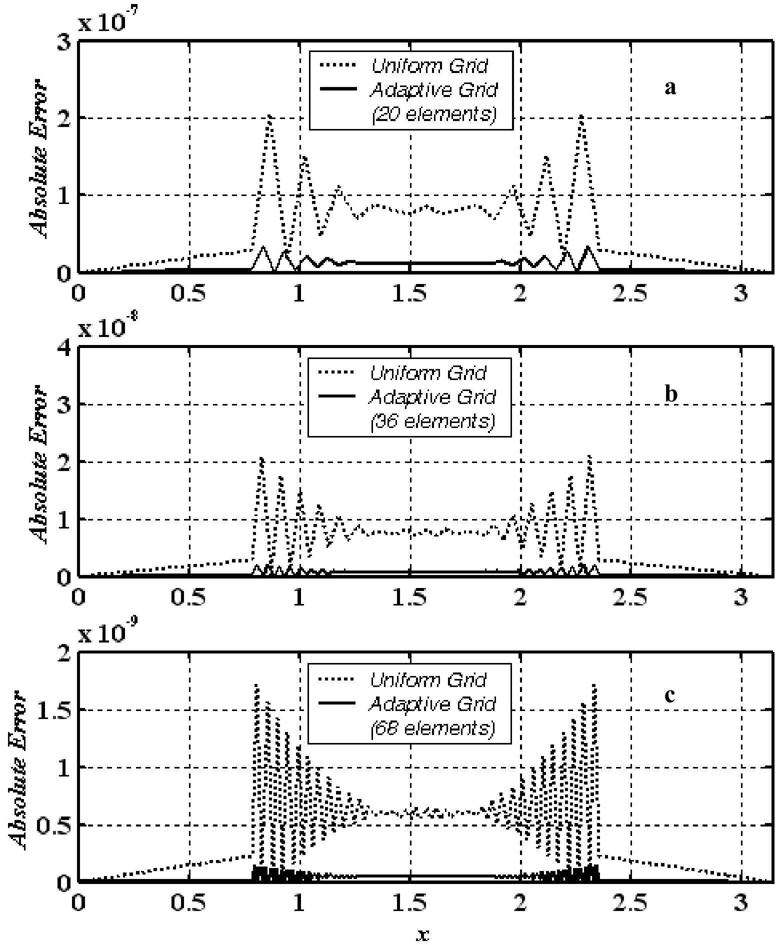

Figure 3: Absolute errors for uniform meshing and adaptive meshing with increasing number of elements (a) 20 elements (b) 36 elements (c) 68 elements.

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