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ANALYSIS OF OPTIMAL ERROR ESTIMATES AND SUPERCONVERGENCE OF THE DISCONTINUOUS GALERKIN METHOD FOR CONVECTION-DIFFUSION PROBLEMS IN ONE SPACE DIMENSION

MAHBOUB BACCOUCH AND HELMI TEMIMI

Abstract. In this paper, we study the convergence and superconvergence properties of the discontinuous Galerkin (DG) method for a linear convection-diffusion problem in one-dimensional setting. We prove that the DG solution and its derivative exhibit optimal $\mathcal{O}(h^{p+1})$ and $\mathcal{O}(h^p)$ convergence rates in the L^2-norm, respectively, when p-degree piecewise polynomials with $p \geq 1$ are used. We further prove that the p-degree DG solution and its derivative are $\mathcal{O}(h^{2p})$ superconvergent at the downwind and upwind points, respectively. Numerical experiments demonstrate that the theoretical rates are optimal and that the DG method does not produce any oscillation. We observed optimal rates of convergence and superconvergence even in the presence of boundary layers when Shishkin meshes are used.

Key words. Discontinuous Galerkin method, convection-diffusion problems, singularly perturbed problems, superconvergence, upwind and downwind points, Shishkin meshes.

1. Introduction

Problems involving convection and diffusion arise in several important applications throughout science and engineering, including fluid flow, heat transfer, among many others. Their typical solutions exhibit boundary and/or interior layers. It is well-known that the standard continuous Galerkin finite element method exhibits poor stability properties for singularly perturbed problems. One of the difficulties in numerically computing the solution of singularly perturbed problems lays in the so-called boundary layer behavior. In the presence of sharp boundary or interior layers, nonphysical oscillations pollute the numerical solution throughout the computational domain. In other words, the solution varies very rapidly in a very thin layer near the boundary. Consult [49, 59, 58, 40, 55, 43] and the references cited therein for a detailed discussion on the topic of singularly perturbed problems. The discontinuous Galerkin (DG) methods have become very popular numerical techniques for solving ordinary and partial differential equations. They have been successfully applied to hyperbolic, elliptic, and parabolic problems arising from a wide range of applications. Over the last years, there has been much interest in applying the DG schemes to problems where the diffusion is not negligible and to convection-diffusion problems.

The DG method considered here is a class of finite element methods using completely discontinuous piecewise polynomials for the numerical solution and the test functions. DG method combines many attractive features of the classical

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finite element and finite volume methods. It is a powerful tool for approximating some differential equations which model problems in physics, especially in fluid dynamics or electrodynamics. Comparing with the standard finite element method, the DG method has a compact formulation, *i.e.*, the solution within each element is weakly connected to neighboring elements. DG method was initially introduced by Reed and Hill in 1973 as a technique to solve neutron transport problems [46]. In 1974, LaSaint and Raviart [42] presented the first numerical analysis of the method for a linear advection equation. Since then, DG methods have been used to solve ordinary differential equations [7, 23, 41, 42], hyperbolic [19, 20, 21, 22, 34, 35, 45, 38, 39, 30, 57, 44, 2, 3, 16, 6] and diffusion and convection-diffusion [17, 18, 53, 36] partial differential equations. The proceedings of Cockburn *et al.* [33] an Shu [51] contain a more complete and current survey of the DG method and its applications.

In recent years, the study of superconvergence of numerical methods has been an active research field in numerical analysis. Superconvergence properties for finite element and DG methods have been extensively studied in [7, 11, 37, 42, 56, 52] for ordinary differential equations, [2, 3, 16, 6, 4, 15, 13, 7, 10] for hyperbolic problems and [14, 5, 9, 10, 16, 24, 27, 30] for diffusion and convection-diffusion problems, just to mention a few citations. A knowledge of superconvergence properties can be used to (i) construct simple and asymptotically exact *a posteriori* estimates of discretization errors and (ii) help detect discontinuities to find elements needing limiting, stabilization and/or refinement. Typically, *a posteriori* error estimators employ the known numerical solution to derive estimates of the actual solution errors. They are also used to steer adaptive schemes where either the mesh is locally refined (h-refinement) or the polynomial degree is raised (p-refinement). For an introduction to the subject of *a posteriori* error estimation see the monograph of Ainsworth and Oden [12].

The first superconvergence result for standard DG solutions of hyperbolic PDEs appeared in Adjerid *et al.* [7]. The authors showed that standard DG solutions of one-dimensional hyperbolic problems using p-degree polynomial approximations exhibit an $\mathcal{O}(h^{p+2})$ superconvergence rate at the roots of $(p+1)$-degree Radau polynomial. They further established a strong $\mathcal{O}(h^{2p+1})$ superconvergence at the downwind end of every element. Recent work on other numerical methods for convection-diffusion and for pure diffusion problems has been reviewed by Cockburn *et al.* [32]. In particular, Baumann and Oden [18] presented a new numerical method which exhibits the best features of both finite volume and finite element techniques. Rivière and Wheeler [47] introduced and analyzed a locally conservative DG formulation for nonlinear parabolic equations. They derived optimal error estimates for the method. Rivière *et al.* [48] analyzed several versions of the Baumann and Oden method for elliptic problems. Wihler and Schwab [54] proved robust exponential rates of convergence of DG methods for stationary convection-diffusion problems in one space dimension. We also mention the work of Castillo, Cockburn, Houston, Süli, Schötzau and Schwab [50, 25, 26] in which optimal a priori error estimates for the hp-version of the local DG (LDG) method for convection-diffusion problems are investigated. Later Adjerid *et al.* [8, 9] investigated the superconvergence of the LDG method applied to diffusion and transient convection-diffusion problems. More recently, Celiker and Cockburn [27] proved a new superconvergence property of a large class of finite element methods for one-dimensional steady state convection-diffusion problems. We also mention the recent work of Shu *et al.*

[29, 30] in which the superconvergence property of the LDG scheme for convection-diffusion equations in one space dimension are proven. Finally, Baccouch [14] analyzed the superconvergence properties of the LDG formulation applied to transient convection-diffusion problems in one space dimension. The author proved that the leading error term on each element for the solution is proportional to a $(p+1)$-degree right Radau polynomial while the leading error term for the solution's derivative is proportional to a $(p+1)$-degree left Radau polynomial, when polynomials of degree at most p are used. He further analyzed the convergence of *a posteriori* error estimates and proved that these error estimates are globally asymptotically exact under mesh refinement.

Cheng and Shu [28] developed a new DG finite element method for solving time dependent partial differential equations with higher order spatial derivatives including the generalized KdV equation, the convection-diffusion equation, and other types of nonlinear equation with fifth order derivatives. Unlike the classical LDG method which was first introduced by Cockburn and Shu in [36] for solving convection-diffusion problems, their method can be applied without introducing any auxiliary variables or rewriting the original equation into a larger system. They designed numerical fluxes to ensure the stability of the schemes. Furthermore, they proved sub-optimal p-th order of accuracy when using piecewise p-th degree polynomials, while computational results show the optimal $(p+1)$-th order of accuracy, under the condition that $p+1$ is greater than or equal to the order of the equation.

In this work, we study the convergence and superconvergence of the DG method applied to a linear convection-diffusion problem (1). We prove that the p-degree DG solution and its derivative exhibit optimal $\mathcal{O}(h^{p+1})$ and $\mathcal{O}(h^p)$ convergence rates in the L^2-norm, respectively. We further prove that the p-degree DG solution and its derivatives are $\mathcal{O}(h^{2p})$ superconvergent at the downwind and upwind points, respectively. Our proofs are valid for arbitrary regular meshes and for P^p polynomials with $p \geq 1$, and for periodic, Dirichlet, and mixed Dirichlet-Neumann boundary conditions. We present several numerical examples to validate the theoretical results. To the best knowledge of the authors, this work presents the first analysis of optimal error estimates and superconvergence at the downwind and upwind points.

This paper is organized as follows: In section 2 we present the DG scheme for solving the convection-diffusion problem and we introduce some notation and definitions. We also present some preliminary results which will be needed in our error analysis. In section 3, we present the DG error analysis and prove our main superconvergence results. In section 4, we present several numerical examples to validate the global superconvergence results. We conclude and discuss our results in section 5.

2. A model problem

In this paper, we study the superconvergence properties for the DG method for solving the following one-dimensional convection-diffusion problem

$$-\epsilon u'' + cu' = f(x), \quad a < x < b, \tag{1a}$$

subject to one of the following three kinds of boundary conditions (mixed Dirichlet-Neumann, purely Dirichlet, and periodic) which are commonly encountered in practice:

$$u(a) = u_l, \qquad u'(b) = u_r, \tag{1b}$$

$$u(a) = u_l, \qquad u(b) = u_r, \tag{1c}$$

$$u(a) = u(b), \qquad u'(a) = u'(b), \tag{1d}$$

where $f(x)$ is a smooth function on $[a, b]$. For the sake of simplicity, we shall consider here only the mixed boundary conditions (1b). This assumption is not essential. If (1c) or (1d) are chosen, the DG method can be easily designed and our results remain true. In this paper, the diffusion constant ϵ is a positive parameter and the velocity c a nonnegative constant. The choice of $c > 0$ guarantees that the location of the boundary layer is at the outflow boundary $x = 1$. In our error analysis, we assume that $\epsilon = \mathcal{O}(1)$. However, our numerical examples indicate that the analysis techniques in this paper is still valid for singularly perturbed problems when Shishkin meshes are used, see section 4.

In order to obtain the weak DG formulation, we divide the computational domain $\overline{\Omega} = [a, b]$ into N subintervals $I_k = [x_{k-1}, x_k],\ k = 1, \ldots, N$, where $a = x_0 < x_1 < \cdots < x_N = b$. We denote the length of I_k by $h_k = x_k - x_{k-1}$. We also denote $h = \max\limits_{1 \le k \le N} h_k$ and $h_{min} = \min\limits_{1 \le k \le N} h_k$ as the length of the largest and smallest subinterval, respectively. Here, we consider regular meshes, that is $h \le \lambda h_{min}$, where $\lambda \ge 1$ is a constant (independent of h) during mesh refinement. If $\lambda = 1$, then the mesh is uniformly distributed. In this case, the nodes and mesh size are defined by

$$x_k = a + k\,h, \quad k = 0, 1, 2, \ldots, N, \quad h = \frac{b-a}{N}.$$

Throughout this paper, we define $v(x_k^-) = \lim\limits_{s \to 0^-} v(x_k + s)$ and $v(x_k^+) = \lim\limits_{s \to 0^+} v(x_k + s)$ to be the left limit and the right limit of the function v at the discontinuity point x_k. We also use $[v](x_k) = v(x_k^+) - v(x_k^-)$ to denote the jump of v at x_k.

The weak DG formulation is obtained by multiplying (1a) on each element I_k by a smooth test function v and integrating over I_k. After integrating by parts we obtain the following weak formulation:

$$\begin{aligned} &(\epsilon u' - cu)(x_{k-1})v(x_{k-1}) - (\epsilon u' - cu)(x_k)v(x_k) - \epsilon u(x_{k-1})v'(x_{k-1}) \\ &+\epsilon u(x_k)v'(x_k) - \int_{I_k} (\epsilon v'' + cv')u dx = \int_{I_k} f v dx. \end{aligned} \tag{2}$$

We define the piecewise-polynomial space V_h^p as the space of polynomials of degree at most p in each subinterval I_k, *i.e.*,

$$V_h^p = \{v : \ v|_{I_k} \in P^p(I_k),\ k = 1, \ldots, N\},$$

where $P^p(I_k)$ is the space of polynomials of degree at most p on I_k. Note that polynomials in the space V_h^p are allowed to have discontinuities across element boundaries.

Next, we approximate the exact solution $u(x)$ by a piecewise polynomial $u_h(x) \in V_h^p$. We note that u_h is not necessarily continuous at the endpoints of I_k. The discrete formulation consists of finding $u_h \in V_h^p$ such that: $\forall\ v \in V_h^p$ and $k =$

$1, \cdots, N$,

$$(\epsilon\hat{u'}_h - c\hat{u}_h)(x_{k-1})v(x_{k-1}^+) - (\epsilon\hat{u'}_h - c\hat{u}_h)(x_k)v(x_k^-) - \epsilon\hat{u}_h(x_{k-1})v'(x_{k-1}^+)$$
$$+\epsilon\hat{u}_h(x_k)v'(x_k^-) - \int_{I_k}(\epsilon v'' + cv')u_h dx = \int_{I_k} fv dx, \tag{3}$$

where the numerical fluxes $\hat{u}_h$ and $\hat{u'}_h$ are, respectively, the discrete approximations to the traces of u and u' at the nodes. In order to complete the definition of the discrete DG method we need to select $\hat{u}_h$ and $\hat{u'}_h$ on the boundaries of I_k. For the mixed boundary conditions (1b), we take the following alternating numerical fluxes; see Cheng and Shu [28]:

$$\hat{u}_h(x_k) = \begin{cases} u_l, & k = 0, \\ u_h(x_k^-), & k = 1, \ldots, N, \end{cases}$$
$$\hat{u'}_h(x_k) = \begin{cases} u_h'(x_k^+), & k = 0, \ldots, N-1, \\ u_r, & k = N. \end{cases} \tag{4a}$$

The numerical fluxes associated with the Dirichlet boundary conditions (1c) can be taken as

$$\hat{u}_h(x_k) = \begin{cases} u_l, & k = 0, \\ u_h(x_k^-), & k = 1, \ldots, N-1, \\ u_r, & k = N, \end{cases}$$
$$\hat{u'}_h(x_k) = \begin{cases} u_h'(x_k^+), & k = 0, \ldots, N-1, \\ u_h'(b^-) + \frac{p}{h_N}(u_h(b^-) - u_r), & k = N. \end{cases} \tag{4b}$$

We note that, if the periodic boundary conditions (1d) are used then the numerical fluxes can be taken as

$$\hat{u}_h(x_k) = u_h(x_k^-), \quad \hat{u'}_h(x_k) = u_h'(x_k^+), \quad k = 0, \ldots, N. \tag{4c}$$

Notation, definitions, and preliminary results. In our analysis we need the pth-degree Legendre polynomial defined by Rodrigues formula [1]

$$\tilde{L}_p(\xi) = \frac{1}{2^p p!}\frac{d^p}{d\xi^p}[(\xi^2-1)^p], \quad -1 \le \xi \le 1,$$

which satisfies the following properties: $\tilde{L}_p(1) = 1$, $\tilde{L}_p(-1) = (-1)^p$, $\tilde{L}_p'(-1) = \frac{p(p+1)}{2}(-1)^{p+1}$, and

$$\int_{-1}^{1}\tilde{L}_k(\xi)\tilde{L}_p(\xi)d\xi = \frac{2}{2k+1}\delta_{kp}, \quad \text{where } \delta_{kp} \text{ is the Kronecker symbol.} \tag{5}$$

Mapping the physical element $I_k = [x_{k-1}, x_k]$ into a reference element $[-1, 1]$ by the standard affine mapping

$$x(\xi, h_k) = \frac{x_k + x_{k-1}}{2} + \frac{h_k}{2}\xi, \tag{6}$$

we obtain the p-degree shifted Legendre polynomial $L_{p,k}(x) = \tilde{L}_p\left(\frac{2x - x_k - x_{k-1}}{h_k}\right)$ on I_k.

In this paper, we define the L^2 inner product of two integrable functions u and v on the interval I_k as $u, v)_{I_k} = \int_{I_k} u(x)v(x)dx$. Denote $\|u\|_{0,I_k} = ((u,u)_{I_k})^{1/2}$ to be the standard L^2-norm of u on I_k. Moreover, the standard L^∞-norm of u on I_k is defined by $\|u\|_{\infty,I_k} = \sup_{x\in I_k}|u(x)|$.

Let $H^s(I_k)$, where $s = 0, 1, \ldots$, denote the standard Sobolev space of square integrable functions on I_k with all derivatives $u^{(j)},\ j = 0, 1, \ldots, s$ being square integrable on I_k *i.e.*,

$$H^s(I_k) = \left\{ u \ : \ \int_{I_k} |u^{(j)}|^2 dx < \infty,\ 0 \le j \le s \right\},$$

and equipped with the norm $\|u\|_{s,I_k} = \left(\sum_{j=0}^{s} \left\| u^{(j)} \right\|_{0,I_k}^2 \right)^{1/2}$. The $H^s(I_k)$-seminorm of a function u on I_k is given by $|u|_{s,I_k} = \left\| u^{(s)} \right\|_{0,I_k}$.

We also define the norms on the whole computational domain Ω as follows:

$$\|u\|_{0,\Omega} = \left(\sum_{k=1}^{N} \|u\|_{0,I_k}^2 \right)^{1/2},\ \|u\|_{s,\Omega} = \left(\sum_{k=1}^{N} \|u\|_{s,I_k}^2 \right)^{1/2},\ \|u\|_{\infty,\Omega} = \max_{1 \le k \le N} \|u\|_{\infty,I_k}.$$

The seminorm on the whole computational domain Ω is defined as $|u|_{s,\Omega} = \left(\sum_{k=1}^{N} |u|_{s,I_k}^2 \right)^{1/2}$. We note that if $u \in H^s(\Omega)$, $s = 1, 2, \ldots$, the norms $\|u\|_{s,\Omega}$ on the whole computational domain is the standard Sobolev norm $\left(\sum_{j=0}^{s} \left\| u^{(j)} \right\|_{0,\Omega}^2 \right)^{1/2}$. For convenience, we use $\|u\|_{I_k}$ and $\|u\|$ to denote $\|u\|_{0,I_k}$ and $\|u\|_{0,\Omega}$, respectively.

For $p \ge 1$, we consider two special projection operators, $P_h^{\pm}$, which are defined as follows: For any smooth function u, the restriction of $P_h^- u$ to I_k is the unique polynomial in $P^p(I_k)$ satisfying

$$\text{(7a)} \quad \int_{I_k} (P_h^- u - u) v dx = 0,\ \forall\ v \in P^{p-1}(I_k), \quad \text{and} \quad (P_h^- u - u)(x_k^-) = 0.$$

Similarly, the restriction of $P_h^+ u$ to I_k is the unique polynomial in $P^p(I_k)$ satisfying

$$\text{(7b)} \quad \int_{I_k} (P_h^+ u - u) v dx = 0,\ \forall\ v \in P^{p-1}(I_k), \quad \text{and} \quad (P_h^+ u - u)(x_{k-1}^+) = 0.$$

These special projections are used in the error estimates of the DG methods to derive optimal L^2 error bounds in the literature, *e.g.*, in [30]. They are mainly used to eliminate the jump terms at the element boundaries in the error estimates in order to prove the optimal L^2 error estimates.

In our analysis, we need the following well-known projection results. The proofs can be found in [31].

Lemma 2.1. *The projections $P_h^{\pm} u$ exist and are unique. Moreover, for any $u \in H^{p+1}(I_k)$ with $k = 1, \cdots, N$, there exists a constant C independent of the mesh size h such that*

$$\left\| (u - P_h^{\pm} u)^{(s)} \right\|_{I_k} \le C h_k^{p+1-s} |u|_{p+1,I_k},$$

$$\text{(8)} \quad \left\| (u - P_h^{\pm} u)^{(s)} \right\| \le C h^{p+1-s} |u|_{p+1,\Omega},\ s = 0, 1, 2.$$

In addition to the projections $P_h^{\pm}$, we also need another projection $\bar{P}_h$ which is defined as follows: For any smooth function u, the restriction of $\bar{P}_h u$ to I_k is the unique polynomial in $P^p(I_k)$ satisfying the following $p+1$ conditions: For $p = 1$, we require the two conditions

$$\text{(9a)} \quad (\bar{P}_h u - u)'(x_{k-1}^+) = 0, \quad (\bar{P}_h u - u)(x_{k-1}^+) = 0.$$

For $p = 2$, we require the two conditions in (9a) and $(\bar{P}_h u - u)(x_k^-) = 0$ *i.e.*,

(9b) $(\bar{P}_h u - u)'(x_{k-1}^+) = 0, \quad (\bar{P}_h u - u)(x_{k-1}^+) = 0, \quad (\bar{P}_h u - u)(x_k^-) = 0,$

and, for $p \geq 3$, we require the three conditions in (9b) and the following $p - 2$ conditions

$$\int_{I_k} (\bar{P}_h u - u) v dx = 0, \ \forall \ v \in P^{p-3}(I_k). \tag{9c}$$

The existence and uniqueness of $\bar{P}_h$ is provided in the following lemma.

Lemma 2.2. *The operator $\bar{P}_h$ exists and unique. Moreover, we have the following* a priori *error estimates: For $p = 1$,*

$$\left|(u - \bar{P}_h u)(x_k^-)\right| \leq h_k^{3/2} |u|_{2,I_k} . \tag{10}$$

Furthermore, for $p \geq 1$,

$$\left\|(u - \bar{P}_h u)^{(s)}\right\|_{I_k} \leq C h_k^{p+1-s} |u|_{p+1,I_k} ,$$

$$\left\|(u - \bar{P}_h u)^{(s)}\right\| \leq C h^{p+1-s} |u|_{p+1,\Omega} , \quad s = 0, 1, 2. \tag{11}$$

Proof. For $p = 1$, $\bar{P}_h u$ is the first-degree Taylor polynomial for u about x_{k-1}. It can be seen from (9a) that $\bar{P}_h u$ is uniquely given by

$$\bar{P}_h u(x) = u(x_{k-1}^+) + u'(x_{k-1}^+)(x - x_{k-1}), \quad x \in I_k.$$

Similarly, for $p = 2$, the three conditions in (9b) give the unique polynomial

$$\begin{aligned}\bar{P}_h u(x) =& u(x_{k-1}^+) + u'(x_{k-1}^+)(x - x_{k-1}) \\ &+ \frac{u(x_k^-) - h_k u'(x_{k-1}^+) - u(x_{k-1}^+)}{h_k^2}(x - x_{k-1})^2, \quad x \in I_k.\end{aligned}$$

Now, we assume $p \geq 3$. We are only going to proof the uniqueness, and since we are working with a linear system of equations, the existence is equivalent to the uniqueness.

Assume that w_1 and w_2 are two polynomials in $P^p(I_k)$ which satisfy (9b) and (9c). Then the difference $w = w_1 - w_2$ satisfies the following $p + 1$ conditions

$$w'(x_{k-1}^+) = 0, \quad w(x_{k-1}^+) = 0, \quad w(x_k^-) = 0, \quad \int_{I_k} w v dx = 0, \quad \forall \ v \in P^{p-3}(I_k). \tag{12}$$

We note the w can be expressed in terms of the Legendre polynomials $w(x) = \sum_{i=0}^{p} c_i L_{i,k}(x)$, $x \in I_k$. Using the orthogonality relation (5) and the $p-2$ conditions $\int_{I_k} w v dx = 0$, $\forall \ v \in P^{p-3}(I_k)$, we get

$$w(x) = c_{p-2} L_{p-2,k}(x) + c_{p-1} L_{p-1,k}(x) + c_p L_{p,k}(x), \quad x \in I_k.$$

Now using the three conditions $w'(x_{k-1}^+) = 0$, $w(x_{k-1}^+) = 0$, $w(x_k^-) = 0$ and the properties of Legendre polynomials $L'_{i,k}(x_{k-1}) = (-1)^{i+1} \frac{i(i+1)}{h_k}$, $L_{i,k}(x_{k-1}) = (-1)^i$, and $L_{i,k}(x_k) = 1$, we get the following linear system of equations

$$\begin{aligned}&(-1)^{p-1} \frac{(p-2)(p-1)}{h_k} c_{p-2} + (-1)^p \frac{p(p-1)}{h_k} c_{p-1} + (-1)^{p+1} \frac{(p+1)p}{h_k} c_p = 0, \\ &(-1)^{p-2} c_{p-2} + (-1)^{p-1} c_{p-1} + (-1)^p c_p = 0, \\ &c_{p-2} + c_{p-1} + c_p = 0.\end{aligned}$$

A direct computation reveals that the determinant of the coefficient matrix of this linear system is $\frac{4(1-2p)}{h_k} \neq 0$. Thus, the system has the trivial solution $c_{p-2} = c_{p-1} = c_p = 0$. Thus $w(x) = 0$ which completes the proof of the existence and uniqueness.

Next, we will prove (10). We note that, for $p = 1$, $\bar{P}_h u$ is the first-degree Taylor polynomial for u about x_{k-1}. Using Taylor's formula with integral remainder, we have

$$u(x) = \bar{P}_h u(x) + \int_{x_{k-1}}^{x} (x_{k-1} - t) u''(t) dt. \tag{13}$$

Thus,

$$\left|(u - \bar{P}_h u)(x_k^-)\right| = \left|\int_{x_{k-1}}^{x_k} (x_{k-1} - t) u''(t) dt\right| \leq \int_{x_{k-1}}^{x_k} |x_{k-1} - t| \, |u''(t)| \, dt.$$

Using $|x_{k-1} - t| \leq h_k, \; x \in I_k$ and applying the Cauchy-Schwarz inequality, we obtain

(14)

$$\left|(u - \bar{P}_h u)(x_k^-)\right| \leq h_k \int_{x_{k-1}}^{x_k} |u''(t)| \, dt \leq h_k^{3/2} \left(\int_{x_{k-1}}^{x_k} |u''(t)|^2 \, dt\right)^{1/2} = h_k^{3/2} \, |u|_{2,I_k} \, .$$

The proof of (11) is similar to that of (8) and is omitted. □

We would like to remark that the operator $\bar{P}_h u$ is introduced only for the purpose of technical proof of error estimates and superconvergence.

Finally, we recall some inverse properties of the finite element space V_h^p which will be used in our error analysis: For any $v_h \in V_h^p$, there exists a positive constant C independent of v_h and h, such that

$$\left\|v_h^{(s)}\right\|_{I_k} \leq C h_k^{-s} \left\|v_h\right\|_{I_k}, \quad s \geq 1, \quad \forall\, k = 1, \ldots, N, \tag{15a}$$

$$\left|v_h(x_{k-1}^+)\right| + \left|v_h(x_k^-)\right| \leq C h_k^{-1/2} \left\|v_h\right\|_{I_k}, \quad \forall\, k = 1, \ldots, N. \tag{15b}$$

From now on, the notation $C, \; C_1, \; C_2,$ etc. will be used to denote positive constants that are independent of the discretization parameters h, but which may depend upon (i) the exact smooth solution of the differential equation (1a) and its derivatives and (ii) the diffusion constant ϵ. Furthermore, all the constants will be generic, *i.e.*, they may represent different constant quantities in different occurrences.

3. Global convergence and superconvergence error analysis

In this section, we investigate the optimal convergence and superconvergence properties of the DG method. We prove that the DG solution and its derivative are $O(h^{2p})$ superconvergent at the downwind and upwind mesh points, respectively. In order to prove these results, we need to derive some error equations.

Throughout this paper, $e = u - u_h$ denotes the error between the exact solution of (1) and the numerical solution defined in (3).

We subtract (3) from (2) with $v \in V_h^p$ and we use the numerical flux (4a) to obtain the DG orthogonality condition for the error e on I_k:

$$\mathcal{A}_k(e;v) = 0, \quad \forall\, v \in V_h^p, \tag{16a}$$

where the bilinear form $\mathcal{A}_k(e;v)$ is given by

$$\begin{aligned}\mathcal{A}_k(e;v) &= (\epsilon e'(x_{k-1}^+) - ce(x_{k-1}^-))v(x_{k-1}^+) - (\epsilon e'(x_k^+) - ce(x_k^-))v(x_k^-)\\ &\quad +\epsilon e(x_k^-)v'(x_k^-) - \epsilon e(x_{k-1}^-)v'(x_{k-1}^+) - \int_{I_k} (\epsilon v'' + cv')e dx.\end{aligned} \tag{16b}$$

Performing a simple integration by parts on the last term of $\mathcal{A}_k(e;v)$ yields

$$\begin{aligned}\mathcal{A}_k(e;v) &= -\epsilon e'(x_k^+)v(x_k^-) + \epsilon e'(x_{k-1}^+)v(x_{k-1}^+)\\ &\quad +[e](x_{k-1})(\epsilon v' + cv)(x_{k-1}^+) + \int_{I_k} (\epsilon v' + cv)e' dx.\end{aligned} \tag{17}$$

Using another integration by parts, we get

$$\mathcal{A}_k(e;v) = -\epsilon[e'](x_k)v(x_k^-) + [e](x_{k-1})(\epsilon v' + cv)(x_{k-1}^+) + \int_{I_k} (-\epsilon e'' + ce')v dx. \tag{18}$$

We note that, with the numerical fluxes (4a), the jumps of e and e' at an interior point x_k are defined as $[e](x_k) = e(x_k^+) - e(x_k^-)$ and $[e'](x_k) = e'(x_k^+) - e'(x_k^-)$. Since $e(x_0^-) = e'(x_N^+) = 0$, the jumps at the endpoints of the computational domain are given by

$$[e](x_0) = e(x_0^+) - e(x_0^-) = e(x_0^+), \quad [e'](x_N) = e'(x_N^+) - e'(x_N^-) = -e'(x_N^-),$$

Next, we state and prove the following results needed for our analysis.

Theorem 3.1. *Let u be the exact solution of (1). Let $p \geq 1$ and u_h be the DG solution of (3) with the numerical fluxes (4a), then there exists a positive constant C which depends on u and ϵ but independent of h such that*

$$\max_{j=1,\ldots,N} \left| e'(x_{j-1}^+) \right| \leq Ch^p \left\| e' \right\|. \tag{19}$$

Proof. We construct the following auxiliary problem: find a function $V \in H^1([x_{j-1}, b])$ such that

$$\epsilon V' + cV = 0, \quad x \in (x_{j-1}, b], \tag{20}$$

$$\text{subject to the boundary condition } V(x_{j-1}) = \frac{1}{\epsilon},$$

where $j = 1, \ldots, N$ is a fixed integer. This problem has the following exact solution

$$V(x) = \frac{1}{\epsilon} \exp\left(-\frac{c}{\epsilon}(x - x_{j-1})\right), \quad x \in \Omega_1 = [x_{j-1}, b]. \tag{21}$$

Using (21), we have the regular estimate $|V|_{p+1,\Omega_1} \leq C$.

On the one hand, taking $v = V$ in (17) and using (20), we have for all $k = j, \ldots, N$

$$\begin{aligned}\mathcal{A}_k(e;V) &= -\epsilon e'(x_k^+)V(x_k^-) + \epsilon e'(x_{k-1}^+)V(x_{k-1}^+)\\ &\quad +[e](x_{k-1})(\epsilon V' + cV)(x_{k-1}^+) + \int_{I_k} (\epsilon V' + cV)e' dx\\ &= -\epsilon e'(x_k^+)V(x_k) + \epsilon e'(x_{k-1}^+)V(x_{k-1}),\end{aligned}$$

which, after summing over $I_k,\ k = j, \dots, N$ and using the fact that $V(x_{j-1}) = 1/\epsilon$ and $e'(x_N^+) = 0$, gives

$$\sum_{k=j}^{N} \mathcal{A}_k(e;V) = -\epsilon e'(x_N^+)V(x_N) + \epsilon e'(x_{j-1}^+)V(x_{j-1}) = e'(x_{j-1}^+). \tag{22}$$

On the other hand, adding and subtracting $\bar{P}_h V$ to V and using (16a) with $v = \bar{P}_h V \in P^p(I_k)$, we write

$$\mathcal{A}_k(e;V) = \mathcal{A}_k(e;V-\bar{P}_hV) + \mathcal{A}_k(e;\bar{P}_hV) = \mathcal{A}_k(e;V-\bar{P}_hV).$$

Using (17) with $v = V - \bar{P}_h V$, we obtain

$$\begin{aligned}\mathcal{A}_k(e;V) &= -\epsilon e'(x_k^+)(V-\bar{P}_hV)(x_k^-) + \epsilon e'(x_{k-1}^+)(V-\bar{P}_hV)(x_{k-1}^+)\\ &\quad + [e](x_{k-1})(\epsilon(V-\bar{P}_hV)' + c(V-\bar{P}_hV))(x_{k-1}^+)\\ &\quad + \int_{I_k} (\epsilon(V-\bar{P}_hV)' + c(V-\bar{P}_hV))e'\,dx.\end{aligned}$$

Applying the properties of the projection $\bar{P}_h$ (9a), we get

$$\mathcal{A}_k(e;V) = -\epsilon e'(x_k^+)(V-\bar{P}_hV)(x_k^-) + \int_{I_k} (\epsilon(V-\bar{P}_hV)' + c(V-\bar{P}_hV))e'\,dx.$$

Summing over the elements $I_k,\ k = j, \dots, N$ and using (22), we obtain

$$\begin{aligned}e'(x_{j-1}^+) &= \sum_{k=j}^{N} \int_{I_k} (\epsilon(V-\bar{P}_hV)' + c(V-\bar{P}_hV))e'\,dx\\ &\quad - \sum_{k=j}^{N} \epsilon e'(x_k^+)(V-\bar{P}_hV)(x_k^-).\end{aligned} \tag{23}$$

We consider the cases $p = 1$ and $p \geq 2$ separately. We first consider the case $p \geq 2$. Since $(V - \bar{P}_h V)(x_k^-) = 0$ by (9b), (23) reduces to

$$e'(x_{j-1}^+) = \sum_{k=j}^{N} \int_{I_k} (\epsilon(V-\bar{P}_hV)' + c(V-\bar{P}_hV))e'\,dx.$$

Applying the Cauchy-Schwarz inequality and the estimate (11) yields

$$\begin{aligned}\left|e'(x_{j-1}^+)\right| &\leq \sum_{k=j}^{N} \int_{I_k} \left(\epsilon\left|(V-\bar{P}_hV)'\right| + c\left|V-\bar{P}_hV\right|\right)\left|e'\right|dx\\ &\leq \left(\epsilon\left\|(V-\bar{P}_hV)'\right\|_{0,\Omega_1} + c\left\|V-\bar{P}_hV\right\|_{0,\Omega_1}\right)\|e'\|_{0,\Omega_1}\\ &\leq \left(\epsilon C_1 h^p\,|V|_{p+1,\Omega_1} + cC_2 h^{p+1}\,|V|_{p+1,\Omega_1}\right)\|e'\| \leq Ch^p\,\|e'\|.\end{aligned}$$

Taking the maximum of both sides, we obtain the estimate (19).

Next, we consider the case $p = 1$. Using the same steps as above and the estimate (10), (23) gives

$$\begin{aligned}
\left|e'(x_{j-1}^+)\right| &\le \sum_{k=j}^{N}\int_{I_k}\left(\epsilon\left|(V-\bar{P}_hV)'\right| + c\left|V-\bar{P}_hV\right|\right)\left|e'\right|dx \\
&\quad + \sum_{k=j}^{N}\epsilon\left|e'(x_k^+)\right|\left|V-\bar{P}_hV)(x_k^-)\right| \\
&\le Ch\left\|e'\right\| + \sum_{k=j}^{N}\epsilon\left|e'(x_k^+)\right|h^{3/2}\left|V\right|_{2,I_k} \le Ch\left\|e'\right\| + Ch^{3/2}\sum_{j=1}^{N}\epsilon\left|e'(x_j^+)\right| \\
&\le Ch\left\|e'\right\| + Ch^{3/2}N\max_{j=1,\dots,N}\left|e'(x_{j-1}^+)\right|,
\end{aligned}$$

since $e'(x_N^+) = 0$. Using the fact that $N \le \frac{b-a}{h_{min}} \le \frac{b-a}{h}$, we get

$$\left|e'(x_{j-1}^+)\right| \le Ch\left\|e'\right\| + Ch^{1/2}\max_{j=1,\dots,N}\left|e'(x_{j-1}^+)\right|.$$

Consequently, $(1 - Ch^{1/2})\max_{j=1,\dots,N}\left|e'(x_{j-1}^+)\right| \le Ch\left\|e'\right\|$. Hence, for all $h^{1/2} \le \frac{1}{2C}$, we have

$$\frac{1}{2}\max_{j=1,\dots,N}\left|e'(x_{j-1}^+)\right| \le (1 - Ch^{1/2})\max_{j=1,\dots,N}\left|e'(x_{j-1}^+)\right| \le Ch\left\|e'\right\|.$$

We conclude that for $p \ge 1$, $\max_{j=1,\dots,N}\left|e'(x_{j-1}^+)\right| \le Ch^p\left\|e'\right\|$, which completes the proof of (19). ☐

Next, we state and prove optimal L^2 error estimate for $\|e'\|$.

Theorem 3.2. *Under the same conditions as in Theorem 3.1, there exists a constant C such that*

$$\|e'\| \le C\ h^p. \tag{24}$$

Proof. We consider the following auxiliary problem: find a function $U \in H^1(\Omega)$ such that

$$\epsilon U' + cU = e', \quad x \in (a,b] \quad \text{subject to} \quad U(a) = 0. \tag{25}$$

The above initial-value problem has a unique solution $U \in H^1(\Omega)$

$$U(x) = \frac{1}{\epsilon}\int_a^x \exp\left(\frac{c}{\epsilon}(s-x)\right)e'(s)ds, \tag{26}$$

that verify the following regular estimates

$$\begin{aligned}
&\|U\| \le C\left\|e'\right\|,\ |U|_{1,\Omega} \le C\left\|e'\right\|, \\
&|U|_{2,\Omega} \le C(\|e'\| + \|e''\|),\ |U(b)| \le C\left\|e'\right\|.
\end{aligned} \tag{27}$$

Taking $v = U$ in (17) and using (25), we obtain

$$\mathcal{A}_k(e;U) = -\epsilon e'(x_k^+)U(x_k) + \epsilon e'(x_{k-1}^+)U(x_{k-1}) + [e](x_{k-1})e'(x_{k-1}^+) + \int_{I_k}(e')^2dx,$$

which, after summing over all elements and using the fact that $U(a) = e'(x_N^+) = 0$, gives

$$\begin{aligned}\sum_{k=1}^{N} \mathcal{A}_k(e;U) &= -\epsilon e'(x_N^+)U(x_N) + \epsilon e'(x_0^+)U(x_0) \\ &\quad + \sum_{k=1}^{N}[e](x_{k-1})e'(x_{k-1}^+) + \sum_{k=1}^{N}\int_{I_k}(e')^2 dx \\ &= \sum_{k=1}^{N}[e](x_{k-1})e'(x_{k-1}^+) + \|e'\|^2 . \end{aligned} \tag{28}$$

Adding and subtracting $\bar{P}_h U$ to U and using (16a) with $v = \bar{P}_h U \in P^p(I_k)$, we get

$$\mathcal{A}_k(e;U) = \mathcal{A}_k(e;U - \bar{P}_h U) + \mathcal{A}_k(e;\bar{P}_h U) = \mathcal{A}_k(e;U - \bar{P}_h U). \tag{29}$$

Applying (17) with $v = U - \bar{P}_h U$ and using the properties of the projection $\bar{P}_h$ (9a), *i.e.*, $(\bar{P}_h u - u)'(x_{k-1}^+) = (\bar{P}_h u - u)(x_{k-1}^+) = 0$, we obtain

$$\begin{aligned}\mathcal{A}_k(e;U) &= -\epsilon e'(x_k^+)(U - \bar{P}_h U)(x_k^-) \\ &\quad + \int_{I_k}(\epsilon(U - \bar{P}_h U)' + c(U - \bar{P}_h U))e' dx. \end{aligned} \tag{30}$$

Summing over all the elements $I_k,\ k = 1,\ldots,N$, we arrive at

$$\begin{aligned}\sum_{k=1}^{N}\mathcal{A}_k(e;U) &= -\epsilon\sum_{k=1}^{N} e'(x_k^+)(U - \bar{P}_h U)(x_k^-) \\ &\quad + \int_a^b(\epsilon(U - \bar{P}_h U)' + c(U - \bar{P}_h U))e' dx. \end{aligned} \tag{31}$$

Combining (28) and (31), we get

$$\|e'\|^2 = T_1 + T_2 + T_3, \tag{32}$$

where

$$T_1 = \int_a^b(\epsilon(U - \bar{P}_h U)' + c(U - \bar{P}_h U))e' dx,$$

$$T_2 = -\epsilon\sum_{k=1}^{N} e'(x_k^+)(U - \bar{P}_h U)(x_k^-),$$

$$T_3 = -\sum_{k=1}^{N}[e](x_{k-1})e'(x_{k-1}^+).$$

Next, we will estimate $T_k,\ k = 1,2,3$ one by one.

Estimate of T_1. Applying the Cauchy-Schwarz inequality and using the estimate (11) yields

$$\begin{aligned} T_1 &\le \epsilon\left(\left\|(U - \bar{P}_h U)'\right\| + c\left\|U - \bar{P}_h U\right\|\right)\|e'\| \\ &\le \epsilon\left(Ch^p\,|u|_{p+1,\Omega} + cCh^{p+1}\,|u|_{p+1,\Omega}\right)\|e'\| \\ &\le C_1 h^p\,\|e'\| . \end{aligned} \tag{33}$$

Estimate of T_2. We consider the cases $p = 1$ and $p \ge 2$ separately. We first consider the case $p \ge 2$. Using the properties of the projection $\bar{P}_h$ (9b), we have

$(\bar{P}_hU - U)(x_k^-) = 0$. Thus, $T_2 = 0$ for $p \geq 2$. Next, we consider the case $p = 1$. Using (19) with $p = 1$, (10), and the regularity estimate (27), we obtain

$$\begin{aligned} T_2 &\leq \epsilon \sum_{k=1}^{N} \left|e'(x_k^+)\right|\left|(U - \bar{P}_hU)(x_k^-)\right| \\ &\leq \sum_{k=1}^{N} (Ch \|e'\|)(h^{3/2} |U|_{2,I_k}) = Ch^{5/2} \|e'\| \sum_{k=1}^{N} |U|_{2,I_k} \\ &\leq Ch^{5/2} \|e'\| N^{1/2} |U|_{2,\Omega} \leq C_3 h^{5/2} N^{1/2} \|e'\| (\|e'\| + \|e''\|) \end{aligned}$$

Since $N \leq \frac{b-a}{h}$, we get

$$T_2 \leq C_3 h^2 \|e'\| (\|e'\| + \|e''\|)$$

Using the smoothness of u, we have $\|e'\| = \|u' - u_h'\| \leq \|u'\| + \|u_h'\| \leq C$ since, for $p = 1$, u_h' is piecewise constant. Furthermore, $\|e''\| = \|u''\| \leq C$ since $u_h'' = 0$ for $p = 1$. We conclude that

$$T_2 \leq C_2 h^2, \quad \text{for } p = 1 \quad \text{and } T_2 = 0, \quad \text{for } p \geq 2. \tag{34}$$

Estimate of T_3. Using the estimate (19), we have

$$T_3 \leq \sum_{k=1}^{N} \left|[e](x_{k-1})\right|\left|e'(x_{k-1}^+)\right| \leq Ch^p \|e'\| \sum_{k=1}^{N} \left|[e](x_{k-1})\right|.$$

Next, we will estimate $\sum_{k=1}^{N} \left|[e](x_{k-1})\right|$. Taking $v = 1$ in (18) and using (16a), we get

$$[e](x_{k-1}) = \epsilon[e'](x_k) + \int_{I_k} (\epsilon e'' - ce')dx = \epsilon e'(x_k^+) - \epsilon e'(x_{k-1}^+) - c \int_{I_k} e'dx.$$

Using the estimate (19) and applying the Cauchy-Schwarz inequality yields

$$\begin{aligned} \left|[e](x_{k-1})\right| &\leq \epsilon\left|e'(x_k^+)\right| + \epsilon\left|e'(x_{k-1}^+)\right| + ch^{1/2} \|e'\|_{0,I_k} \\ &\leq \epsilon Ch^p \|e'\| + \epsilon Ch^p \|e'\| + ch^{1/2} \|e'\|_{0,I_k} \leq C_1 h^p \|e'\| + ch^{1/2} \|e'\|_{0,I_k}. \end{aligned}$$

Summing over all elements, applying the Cauchy-Schwarz inequality, and using the fact that $N \leq \frac{b-a}{h_{min}} \leq \frac{b-a}{h}$, we get

$$\begin{aligned} \sum_{k=1}^{N} \left|[e](x_{k-1})\right| &\leq C_1 N h^p \|e'\| + ch^{1/2} \sum_{k=1}^{N} \|e'\|_{0,I_k} \\ &\leq C_1 N h^p \|e'\| + ch^{1/2} N^{1/2} \|e'\| \\ &\leq C_1 (b-a) h^{p-1} \|e'\| + c(b-a)^{1/2} \|e'\| \\ &\leq C_2 h^{p-1} \|e'\| + C_3 \|e'\| \leq C_4 \|e'\|, \quad p \geq 1. \end{aligned}$$

Therefore, we conclude that

$$T_3 \leq C_3 h^p \|e'\|^2. \tag{35}$$

Now, combining (32) with (33), (34), (35) and applying the inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$, we get, for $p \geq 2$,

$$\|e'\|^2 \leq C_1 h^p \|e'\| + C_3 h^p \|e'\|^2 \leq \frac{1}{2} C_1^2 h^{2p} + \frac{1}{2} \|e'\|^2 + C_3 h^p \|e'\|^2,$$

which gives $\|e'\|^2 \leq C_1^2 h^{2p} + 2C_3 h^p \|e'\|^2$. Similarly, if $p = 1$, we have an extra term $T_2 \leq C_2 h^2$. Thus, for all $p \geq 1$, we have

$$\|e'\|^2 \leq Ch^{2p} + Ch^p \|e'\|^2 \leq Ch^{2p} + Ch \|e'\|^2 ,$$

Hence, for all $h \leq \frac{1}{2C}$, we have $\frac{1}{2} \|e'\|^2 \leq (1 - Ch) \|e'\|^2 \leq Ch^{2p}$, which completes the proof of (24). □

In the following corollary, we state and prove $2p$-order superconvergence of the solution's derivative at the upwind points.

Corollary 3.1. *Under the same conditions as in Theorem 3.1, there exists a constant C such that*

$$\max_{j=1,\ldots,N} \left|e'(x_{j-1}^+)\right| \leq Ch^{2p}. \tag{36}$$

Proof. Combining the estimates (19) and (24), we immediately obtain (36). □

Next, we state and prove optimal L^2 error estimates for $\|e\|$.

Theorem 3.3. *Under the same conditions as in Theorem 3.1, there exists a constant C such that*

$$\|e\| \leq C\, h^{p+1}. \tag{37}$$

Proof. The main idea behind the proof of (37) is to construct the following adjoint problem: find a function W such that

$$\begin{aligned} -\epsilon W'' - cW' = e, \quad x \in (a,b), \\ \text{subject to} \quad W(a) = 0, \quad \epsilon W'(b) + cW(b) = 0. \end{aligned} \tag{38}$$

This boundary-value problem has the following exact solution

$$\begin{aligned} W(x) &= -\frac{1}{\epsilon} \int_a^x \left(\int_b^y e(s) \exp\left(\frac{c}{\epsilon}(s-y)\right) ds \right) dy \\ &\quad + \frac{1 - e^{-\frac{c}{\epsilon}(x-a)}}{\epsilon} \int_a^b \left(\int_b^y e(s) \exp\left(\frac{c}{\epsilon}(s-y)\right) ds \right) dy, \end{aligned}$$

that satisfy the regular estimate

$$|W|_{2,\Omega} \leq C \|e\| . \tag{39}$$

On the one hand, taking $v = W$ in (16b) and using (38), we obtain

$$\begin{aligned} \mathcal{A}_k(e;W) &= (\epsilon e'(x_{k-1}^+) - ce(x_{k-1}^-))W(x_{k-1}) - (\epsilon e'(x_k^+) - ce(x_k^-))W(x_k) \\ &\quad + \epsilon e(x_k^-)W'(x_k^-) - \epsilon e(x_{k-1}^-)W'(x_{k-1}^+) - \int_{I_k} (\epsilon W'' + cW')e dx \\ &= (\epsilon e'(x_{k-1}^+) - ce(x_{k-1}^-))W(x_{k-1}) - (\epsilon e'(x_k^+) - ce(x_k^-))W(x_k) \\ &\quad + \epsilon e(x_k^-)W'(x_k) - \epsilon e(x_{k-1}^-)W'(x_{k-1}) + \int_{I_k} e^2 dx. \end{aligned} \tag{40}$$

Summing over all elements and using $W(x_0) = e(x_0^-) = e'(x_N^+) = 0$ and $\epsilon W'(x_N) + cW(x_N) = 0$, gives

$$\begin{aligned}\sum_{k=1}^{N} \mathcal{A}_k(e;W) &= (\epsilon e'(x_0^+) - ce(x_0^-))W(x_0) - (\epsilon e'(x_N^+) - ce(x_N^-))W(x_N) \\ &\quad + \epsilon e(x_N^-)W'(x_N) - \epsilon e(x_0^-)W'(x_0) + \|e\|^2 \\ &= e(x_N^-)\left(\epsilon W'(x_N) + cW(x_N)\right) + \|e\|^2 = \|e\|^2 . \end{aligned} \tag{41}$$

On the other hand, adding and subtracting $\bar{P}_h W$ to W and applying (16a) with $v = \bar{P}_h W \in P^p(I_k)$ yields

$$\mathcal{A}_k(e;W) = \mathcal{A}_k(e;W - \bar{P}_h W) + \mathcal{A}_k(e;\bar{P}_h W) = \mathcal{A}_k(e;W - \bar{P}_h W). \tag{42}$$

We consider the cases $p = 1$ and $p \geq 2$ separately. We first consider the case $p \geq 2$. Using (17) and the property of the projection $\bar{P}_h$, (42) gives

$$\mathcal{A}_k(e;W) = \int_{I_k} \left(\epsilon(W - \bar{P}_h W)' + c(W - \bar{P}_h W)\right) e' dx, \tag{43}$$

which, after summing over all elements and applying the Cauchy-Schwarz inequality yields

$$\begin{aligned}\sum_{k=1}^{N} \mathcal{A}_k(e;W) &= \sum_{k=1}^{N} \int_{I_k} (\epsilon(W - \bar{P}_h W)' + c(W - \bar{P}_h W))e' dx \\ &\leq \left(\epsilon \left\|(W - \bar{P}_h W)'\right\| + c \left\|W - \bar{P}_h W\right\|\right) \|e'\| \end{aligned}$$

Applying the standard interpolation error estimate (11), the regularity estimate (39), and the estimate (24), we get

$$\begin{aligned}\sum_{k=1}^{N} \mathcal{A}_k(e;W) &\leq (\epsilon C_1 h |W|_{2,\Omega} + cC_2 h^2 |W|_{2,\Omega})C_3 h^p \\ &\leq (\epsilon C_1 h C_4 \|e\| + cC_2 h^2 C_4 \|e\|)C_3 h^p \\ &\leq C(h^{p+1} + h^{p+2}) \|e\| \leq Ch^{p+1} \|e\| . \end{aligned} \tag{44}$$

Combining the two formulas (41) and (44) yields $\|e\|^2 \leq Ch^{p+1} \|e\|$, which completes the proof of (37) in the case $p \geq 2$.

Next, we consider the case $p = 1$. We note that (41) is still valid. However (43) is not since $\bar{P}_h W$ is defined by the two conditions (9a). Thus, we proceed differently. Adding and subtracting $\bar{P}_h W$ to W, using (16a), (17), and the properties of the operator $\bar{P}_h$ (9a), we obtain

$$\begin{aligned}\mathcal{A}_k(e;W) &= \mathcal{A}_k(e;W - \bar{P}_h W) \\ &= -\epsilon e'(x_k^+)(W - \bar{P}_h W)(x_k^-) + \int_{I_k} (\epsilon(W - \bar{P}_h W)' + c(W - \bar{P}_h W))e' dx. \end{aligned} \tag{45}$$

Summing over all the elements $I_k,\ k=1,\dots,N$ and using (41), we get

$$\begin{aligned}\|e\|^2 &= -\sum_{k=1}^{N}\epsilon e'(x_k^+)(W-\bar{P}_hW)(x_k^-)\\ &\quad+\sum_{k=1}^{N}\int_{I_k}(\epsilon(W-\bar{P}_hW)'+c(W-\bar{P}_hW))e'dx\\ &\le \sum_{k=1}^{N}\epsilon|e'(x_k^+)|\left|(W-\bar{P}_hW)(x_k^-)\right|\\ &\quad+\sum_{k=1}^{N}\int_{I_k}\left(\epsilon\left|(W-\bar{P}_hW)'\right|+c\left|W-\bar{P}_hW\right|\right)|e'|dx.\end{aligned}$$

Applying the Cauchy-Schwarz inequality, using the estimate (36) with $p=1$, and using the standard interpolation error estimate (10) *i.e.*, $\left|(W-\bar{P}_hW)(x_k^-)\right|\le h_k^{3/2}\,|W|_{2,I_k},\quad k=1,\dots,N$, we obtain

$$\begin{aligned}\|e\|^2 &\le \sum_{k=1}^{N}\epsilon(C_1h^2)(h_k^{3/2}\,|W|_{2,I_k})+\left(\epsilon\left\|(W-\bar{P}_hW)'\right\|+c\left\|W-\bar{P}_hW\right\|\right)\|e'\|\\ &\le C_2\epsilon h^{7/2}\sum_{k=1}^{N}|W|_{2,I_k}+\left(\epsilon\left\|(W-\bar{P}_hW)'\right\|+c\left\|W-\bar{P}_hW\right\|\right)\|e'\|.\end{aligned}$$

Applying the Cauchy-Schwarz inequality $\sum_{k=1}^{N}a_kb_k\le\left(\sum_{k=1}^{N}a_k^2\right)^{1/2}\left(\sum_{k=1}^{N}b_k^2\right)^{1/2}$, the error estimates (11), (24), and the regular estimate (39), we get

$$\begin{aligned}\|e\|^2 &\le C_2\epsilon h^{7/2}\left(\frac{b-a}{h}\right)^{1/2}\left(\sum_{k=1}^{N}|W|_{2,I_k}^2\right)^{1/2}\\ &\quad+\left(\epsilon C_3h\,|W|_{2,\Omega}+cC_4h^2\,|W|_{2,\Omega}\right)C_5h\\ &= \left(C_2(b-a)\epsilon h^3+C_5\left(\epsilon C_3h^2+cC_4h^3\right)\right)|W|_{2,\Omega}\\ &\le \left(C_2(b-a)\epsilon h^3+C_5\left(\epsilon C_3h^2+cC_4h^3\right)\right)C_6\,\|e\|\\ &\le C\,(1+h)\,h^2\,\|e\|,\end{aligned}\tag{46}$$

Thus, we get $\|e\|\le C\,(1+h)\,h^2=\mathcal{O}(h^2)$, which completes the proof of (37) in the case $p=1$. □

Finally, we state and prove $2p$-order superconvergence of the solution at the downwind points.

Theorem 3.4. *Under the same conditions as in Theorem 3.1, there exists a constant C such that*

$$\left|e(x_j^-)\right| \le Ch^{2p},\quad j=1,\dots,N.\tag{47}$$

Proof. Again, the main idea behind the proof of (47) is to construct the following auxiliary problem: find a function $\varphi\in H^2([a,x_j])$ such that

$$-\epsilon\varphi''-c\varphi'=0,\quad x\in[a,x_j),\quad \text{subject to}\ \ \varphi(x_j)=0,\ \varphi'(x_j)=\frac{1}{\epsilon},\tag{48}$$

where $j = 1, \dots, N$ is a fixed integer. This problem has the following exact solution

$$\varphi(x) = \frac{1}{c}\left(1 - \exp\left(\frac{c}{\epsilon}(x_j - x)\right)\right), \quad x \in \Omega_2 = [a, x_j]. \tag{49}$$

Using (49), we can easily show the following estimates

$$|\varphi(a)| = \frac{1}{c}\left(\exp\left(\frac{c}{\epsilon}(x_j - a)\right) - 1\right) = C > 0, \qquad |\varphi|_{p+1,\Omega_2} \le C. \tag{50}$$

We follow the reasoning of Adjerid *et al.* [11, Theorem 2.2]. First we prove (47) for the case $p \ge 3$. For $p \ge 3$, let us consider an interpolation operator I_h which is defined as follows: For any smooth function φ, $I_h\varphi \in V_h^p \cap H^2(\Omega)$ and the restriction of $I_h\varphi$ to I_k is the unique polynomial in $P^p(I_k)$ satisfying: for each $k = 1, \dots, j$,

$$I_h\varphi(x_{k-1}^+) = \varphi(x_{k-1}^+), \quad (I_h\varphi)'(x_{k-1}^+) = \varphi'(x_{k-1}^+),$$
$$I_h\varphi(x_k^-) = \varphi(x_k^-), \quad (I_h\varphi)'(x_k^-) = \varphi'(x_k^-),$$

and $I_h\varphi$ interpolates φ at $p-3$ additional distinct points $\bar{x}_k$, $k = 1, \dots, p-3$ in (x_{k-1}, x_k). In our analysis, we need the following a priori error estimate [31]: For any $\varphi \in H^{p+1}(\Omega_2)$, there exists a constant C independent of the mesh size h such that

$$\|\varphi - I_h\varphi\|_{s,\Omega_2} \le Ch^{p+1-s} |\varphi|_{p+1,\Omega_2}, \quad s = 0, 1, 2. \tag{51}$$

On the one hand, taking $v = \varphi$ in (16b) and using (48), we obtain

$$\begin{aligned}
\mathcal{A}_k(e;\varphi) &= (\epsilon e'(x_{k-1}^+) - ce(x_{k-1}^-))\varphi(x_{k-1}^+) - (\epsilon e'(x_k^+) - ce(x_k^-))\varphi(x_k^-) \\
&\quad + \epsilon e(x_k^-)\varphi'(x_k^-) - \epsilon e(x_{k-1}^-)\varphi'(x_{k-1}^+) - \int_{I_k} (\epsilon\varphi'' + c\varphi')e dx \\
&= (\epsilon e'(x_{k-1}^+) - ce(x_{k-1}^-))\varphi(x_{k-1}) - (\epsilon e'(x_k^+) - ce(x_k^-))\varphi(x_k) \\
&\quad + \epsilon e(x_k^-)\varphi'(x_k) - \epsilon e(x_{k-1}^-)\varphi'(x_{k-1}).
\end{aligned} \tag{52}$$

Summing over all the elements I_k, $k = 1, \dots, j$ and using $\varphi(x_j) = e(x_0^-) = 0$, and $\varphi'(x_j) = 1/\epsilon$, gives

$$\begin{aligned}
\sum_{k=1}^{j} \mathcal{A}_k(e;\varphi) &= (\epsilon e'(x_0^+) - ce(x_0^-))\varphi(x_0) - (\epsilon e'(x_j^+) - ce(x_j^-))\varphi(x_j) \\
&\quad + \epsilon e(x_j^-)\varphi'(x_j) - \epsilon e(x_0^-)\varphi'(x_0) \\
&= \epsilon e'(x_0^+)\varphi(x_0) + e(x_j^-).
\end{aligned} \tag{53}$$

On the other hand, adding and subtracting $I_h\varphi$ to φ and using (16a), we write $\mathcal{A}_k(e;\varphi)$ as

$$\mathcal{A}_k(e;\varphi) = \mathcal{A}_k(e;\varphi - I_h\varphi) + \mathcal{A}_k(e;\varphi) = \mathcal{A}_k(e;\varphi - I_h\varphi). \tag{54}$$

Using (16b) and the properties of the operator I_h, we obtain

$$\begin{aligned}
\mathcal{A}_k(e;\varphi) &= (\epsilon e'(x_{k-1}^+) - ce(x_{k-1}^-))(\varphi - I_h\varphi)(x_{k-1}^+) \\
&\quad - (\epsilon e'(x_k^+) - ce(x_k^-))(\varphi - I_h\varphi)(x_k^-) \\
&\quad + \epsilon e(x_k^-)(\varphi - I_h\varphi)'(x_k^-) - \epsilon e(x_{k-1}^-)(\varphi - I_h\varphi)'(x_{k-1}^+) \\
&\quad - \int_{I_k} (\epsilon(\varphi - I_h\varphi)'' + c(\varphi - I_h\varphi)')e dx \\
&= - \int_{I_k} (\epsilon(\varphi - I_h\varphi)'' + c(\varphi - I_h\varphi)')e dx.
\end{aligned} \tag{55}$$

Summing over all the elements $I_k,\ k=1,\ldots,j$, we get

$$\sum_{k=1}^{j}\mathcal{A}_k(e;\varphi) \;=\; -\sum_{k=1}^{j}\int_{I_k}(\epsilon(\varphi-I_h\varphi)''+c(\varphi-I_h\varphi)')e\,dx. \tag{56}$$

Combining the two formulas (53) and (56) yields

$$\epsilon e'(x_0^+)\varphi(x_0)+e(x_j^-) \;=\; -\sum_{k=1}^{j}\int_{I_k}(\epsilon(\varphi-I_h\varphi)''+c(\varphi-I_h\varphi)')e\,dx. \tag{57}$$

Using the estimates (50) and (36) and applying the Cauchy-Schwarz inequality yields

$$\begin{aligned}\left|e(x_j^-)\right| &\le \epsilon\left|e'(x_0^+)\right||\varphi(x_0)|+\sum_{k=1}^{j}\int_{I_k}(\epsilon\left|(\varphi-I_h\varphi)''\right|+c\left|(\varphi-I_h\varphi)'\right|)\left|e\right|dx\\ &\le \epsilon(C_0h^{2p})(C_1)+(\epsilon\left\|(\varphi-I_h\varphi)''\right\|_{0,\Omega_2}+c\left\|(\varphi-I_h\varphi)'\right\|_{0,\Omega_2})\left\|e\right\|_{0,\Omega_2}\\ &\le \epsilon C_0C_1h^{2p}+(\epsilon\left\|(\varphi-I_h\varphi)''\right\|_{0,\Omega_2}+c\left\|(\varphi-I_h\varphi)'\right\|_{0,\Omega_2})\left\|e\right\|. \end{aligned}\tag{58}$$

Applying the standard interpolation error estimate (51) and the estimate (37), we get

$$\begin{aligned}\left|e(x_j^-)\right| \le\ & \epsilon C_0C_1h^{2p}+(\epsilon C_2h^{p-1}\left|\varphi\right|_{p+1,\Omega}\\ &+cC_3h^{p}\left|\varphi\right|_{p+1,\Omega})C_4h^{p+1}=\mathcal{O}(h^{2p}),\end{aligned}\tag{59}$$

which completes the proof of (47) when $p\ge 3$.

For $p=2$, let us consider another interpolation operator T_h which is defined as follows: For any smooth function φ, $T_h\varphi\in V_h^2$ and the restriction of $T_h\varphi$ to I_k is the unique polynomial in $P^2(I_k)$ satisfying: for each $k=1,\cdots,j$,

$$T_h\varphi(x_{k-1}^+)=\varphi(x_{k-1}^+),\quad T_h\varphi(x_k^-)=\varphi(x_k^-),\quad (T_h\varphi)'(x_k^-)=\varphi'(x_k^-).$$

In our analysis, we need the following a priori error estimate [31]: For any $\varphi\in H^3(\Omega_2)$, there exists a constant C independent of the mesh size h such that

$$\left\|\varphi-T_h\varphi\right\|_{s,\Omega_2}\le Ch^{3-s}\left\|\varphi\right\|_{3,\Omega_2},\quad s=0,1,2. \tag{60}$$

Adding and subtracting $T_h\varphi$ to φ, using (16a), and the properties of the operator T_h, we obtain

$$\begin{aligned}\mathcal{A}_k(e;\varphi) &= \mathcal{A}_k(e;\varphi-T_h\varphi)\\ &= (\epsilon e'(x_{k-1}^+)-ce(x_{k-1}^-))(\varphi-T_h\varphi)(x_{k-1}^+)\\ &\quad -(\epsilon e'(x_k^+)-ce(x_k^-))(\varphi-T_h\varphi)(x_k^-)\\ &\quad +\epsilon e(x_k^-)(\varphi-T_h\varphi)'(x_k^-)-\epsilon e(x_{k-1}^-)(\varphi-T_h\varphi)'(x_{k-1}^+)\\ &\quad -\int_{I_k}(\epsilon(\varphi-T_h\varphi)''+c(\varphi-T_h\varphi)')e\,dx\\ &= -\epsilon e(x_{k-1}^-)(\varphi-T_h\varphi)'(x_{k-1}^+)-\int_{I_k}(\epsilon(\varphi-T_h\varphi)''+c(\varphi-T_h\varphi)')e\,dx.\end{aligned}\tag{61}$$

Summing over all the elements $I_k,\ k=1,\dots,j$ and using (53), we arrive at

$$\begin{aligned}\epsilon e'(x_0^+)\varphi(x_0)+e(x_j^-) &= -\sum_{k=1}^{j}\epsilon e(x_{k-1}^-)(\varphi-T_h\varphi)'(x_{k-1}^+)\\ &\quad -\sum_{k=1}^{j}\int_{I_k}(\epsilon(\varphi-T_h\varphi)''+c(\varphi-T_h\varphi)')e dx.\end{aligned}$$

Applying the Cauchy-Schwarz inequality and using the estimates (50), (36), and the standard interpolation error estimates $\left|(\varphi-T_h\varphi)'(x_{k-1}^+)\right|\le Ch_k^2\left\|\varphi^{(3)}\right\|_{\infty,I_k},\quad k=1,\dots,j$, we obtain

$$\begin{aligned}\left|e(x_j^-)\right| &\le \epsilon\left|e'(x_0^+)\right||\varphi(x_0)|+\sum_{k=1}^{j}\epsilon\left|e(x_{k-1}^-)\right|\left|(\varphi-T_h\varphi)'(x_{k-1}^+)\right|+\\ &\quad \sum_{k=1}^{j}\int_{I_k}(\epsilon\left|(\varphi-T_h\varphi)''\right|+c\left|(\varphi-T_h\varphi)'\right|)|e|\,dx\\ &\le \epsilon(C_0h^4)(C_1)+C_2\epsilon h^2\left\|\varphi^{(3)}\right\|_{\infty,\Omega_2}\sum_{k=1}^{j}\left|e(x_{k-1}^-)\right|+\\ &\quad (\epsilon\left\|(\varphi-T_h\varphi)''\right\|_{0,\Omega_2}+c\left\|(\varphi-T_h\varphi)'\right\|_{0,\Omega_2})\left\|e\right\|_{0,\Omega_2}.\end{aligned}\tag{62}$$

Applying the standard interpolation error estimate (60) and the estimate (37), we get

$$\begin{aligned}\left|e(x_j^-)\right| &\le \epsilon C_0C_1h^4+C_2\epsilon h^2\left\|\varphi^{(3)}\right\|_{\infty,\Omega_2}\sum_{k=1}^{j}\left|e(x_{k-1}^-)\right|\\ &\quad +(\epsilon C_2h\left|\varphi\right|_{3,\Omega}+cC_3h^2\left|\varphi\right|_{3,\Omega})C_4h^3\\ &\le C\left(h^4+h^2\sum_{k=1}^{j}\left|e(x_{k-1}^-)\right|\right).\end{aligned}\tag{63}$$

Next, we use induction and (63) to prove $\left|e(x_j^-)\right|\le Ch^4$. Taking $v=1$ in (16b) with $k=1$, using (16a) and $e(x_0^-)=0$, we get

$$\begin{aligned}0=\mathcal{A}_1(e;1) &= \epsilon e'(x_0^+)-ce(x_0^-)-\epsilon e'(x_1^+)+ce(x_1^-)\\ &= \epsilon e'(x_0^+)-\epsilon e'(x_1^+)+ce(x_1^-).\end{aligned}\tag{64}$$

Applying (36) with $p=2$, we obtain

$$\begin{aligned}\left|e(x_1^-)\right|=\left|\frac{\epsilon}{c}(e'(x_1^+)-e'(x_0^+))\right| &\le \frac{\epsilon}{c}(\left|e'(x_1^+)\right|+\left|e'(x_0^+)\right|)\\ &\le \frac{\epsilon}{c}(C_0h^4+C_1h^4)=Ch^4.\end{aligned}\tag{65}$$

Next, if we assume that $\left|e(x_k^-)\right|\le C_1h^4,\ k<j$, then (63) yields

$$\begin{aligned}\left|e(x_j^-)\right| &\le C\left(h^4+h^2\sum_{k=1}^{j}\left|e(x_{k-1}^-)\right|\right)\le C\left(h^4+h^2\sum_{k=1}^{j}C_1h^4\right)\\ &\le C\left(h^4+C_1h^5\right)=\mathcal{O}(h^4),\end{aligned}$$

which completes the proof of (47) for the case $p=2$.

Finally, we consider the case $p=1$. We introduce an interpolation operator L_h which is defined as follows: For any smooth function φ, $L_h\varphi \in V_h^1$ and the restriction of $L_h\varphi$ to I_k is the unique polynomial in $P^1(I_k)$ satisfying: for each $k=1,\dots,j$,

$$L_h\varphi(x_{k-1}^+) = \varphi(x_{k-1}^+), \quad L_h\varphi(x_k^-) = \varphi(x_k^-).$$

The following *a priori* error estimate [31] holds: For any $\varphi \in H^2(\Omega_2)$, there exists a constant C independent of the mesh size h such that

$$\|\varphi - L_h\varphi\|_{s,\Omega_2} \le Ch^{2-s}\|\varphi\|_{2,\Omega_2}, \quad s=0,1. \tag{66}$$

Adding and subtracting $L_h\varphi$ to φ, using (16a), (16b), and the properties of the operator L_h, we obtain

$$\begin{aligned}\mathcal{A}_k(e;\varphi) &= \mathcal{A}_k(e;\varphi - L_h\varphi)\\ &= (\epsilon e'(x_{k-1}^+) - ce(x_{k-1}^-))(\varphi - L_h\varphi)(x_{k-1}^+)\\ &\quad -(\epsilon e'(x_k^+) - ce(x_k^-))(\varphi - L_h\varphi)(x_k^-)\\ &\quad +\epsilon e(x_k^-)(\varphi - L_h\varphi)'(x_k^-) - \epsilon e(x_{k-1}^-)(\varphi - L_h\varphi)'(x_{k-1}^+)\\ &\quad -\int_{I_k}(\epsilon(\varphi - L_h\varphi)'' + c(\varphi - L_h\varphi)')e\,dx\\ &= \epsilon e(x_k^-)(\varphi - L_h\varphi)'(x_k^-)\\ &\quad -\epsilon e(x_{k-1}^-)(\varphi - L_h\varphi)'(x_{k-1}^+) - \int_{I_k}(\epsilon(\varphi - L_h\varphi)'' + c(\varphi - L_h\varphi)')e\,dx.\end{aligned} \tag{67}$$

Since $L_h\varphi$ is a linear function on I_k, we have $(L_h\varphi)''(x)=0$. Thus, (67) simplifies to

$$\begin{aligned}\mathcal{A}_k(e;\varphi) &= \epsilon e(x_k^-)(\varphi - L_h\varphi)'(x_k^-) - \epsilon e(x_{k-1}^-)(\varphi - L_h\varphi)'(x_{k-1}^+)\\ &\quad -\int_{I_k}(\epsilon\varphi'' + c(\varphi - L_h\varphi)')e\,dx.\end{aligned}$$

Summing over all the elements $I_k,\ k=1,\dots,j$, using $e(x_0^-)=0$ and (53) we arrive at

$$\begin{aligned}\epsilon e'(x_0^+)\varphi(x_0) + e(x_j^-) &= \epsilon e(x_j^-)(\varphi - L_h\varphi)'(x_j^-) - \epsilon e(x_0^-)(\varphi - L_h\varphi)'(x_0^+) -\\ &\quad \sum_{k=1}^{j}\int_{I_k}(\epsilon\varphi'' + c(\varphi - L_h\varphi)')e\,dx\\ &= \epsilon e(x_j^-)(\varphi - L_h\varphi)'(x_j^-) - \sum_{k=1}^{j}\int_{I_k}(\epsilon\varphi'' + c(\varphi - L_h\varphi)')e\,dx.\end{aligned}$$

Applying the Cauchy-Schwarz inequality and using the estimates (50), (36), and the standard interpolation error estimates $\left|(\varphi - L_h\varphi)'(x_{k-1}^+)\right| \le Ch_k\|\varphi''\|_{\infty,I_k}, \quad k=1,\dots,j$, we obtain

$$\begin{aligned}\left|e(x_j^-)\right| &\le \epsilon\left|e'(x_0^+)\right||\varphi(x_0)| + \epsilon\left|e(x_j^-)\right|\left|(\varphi - L_h\varphi)'(x_j^-)\right|\\ &\quad +\sum_{k=1}^{j}\int_{I_k}(\epsilon|\varphi''| + c|(\varphi - L_h\varphi)'|)|e|\,dx\\ &\le \epsilon(C_0h^2)(C_1) + C_2\epsilon h\|\varphi''\|_{\infty,I_j}\left|e(x_j^-)\right|\\ &\quad +(\epsilon\|\varphi''\|_{0,\Omega_2} + c\|(\varphi - L_h\varphi)'\|_{0,\Omega_2})\|e\|_{0,\Omega_2}.\end{aligned}$$

Applying the standard interpolation error estimate (66) and the estimate (37), we get

$$\begin{aligned} \left|e(x_j^-)\right| &\leq \epsilon C_0 C_1 h^2 + C_2 \epsilon h \left\|\varphi''\right\|_{\infty,\Omega_2} \left|e(x_j^-)\right| + (\epsilon \left\|\varphi''\right\|_{0,\Omega} + c C_3 h \left|\varphi\right|_{2,\Omega}) C_4 h^2 \\ &\leq C\left(h^2 + h\left|e(x_j^-)\right|\right), \end{aligned} \quad (68)$$

which gives $(1 - Ch)\left|e(x_j^-)\right| \leq Ch^2$. Therefore, for small h, $\left|e(x_j^-)\right| = \mathcal{O}(h^2)$. Thus, we have completed the proof of (47) for the case $p = 1$. WE conclude that the superconvergence result (47) is valid for all $p \geq 1$. □

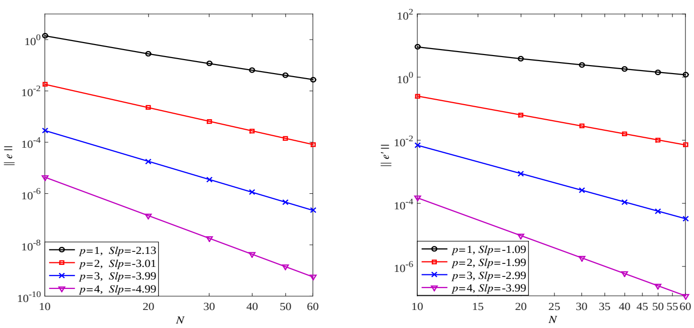

Figure 1: The L^2-norm of the error $\|e\|$ (left) and the derivative of error $\|e'\|$ (right) for Example 4.1 versus N using $p = 1, 2, 3, 4$.

TABLE 1. Maximum errors $\|e\|_\infty^*$ at the downwind points for Example 4.1 using p =1, 2, 3 and 4.

N	$p=1$		$p=2$		$p=3$		$p=4$	
	$\|e\|_\infty^*$	Order	$\|e\|_\infty^*$	Order	$\|e\|_\infty^*$	Order	$\|e\|_\infty^*$	Order
18	3.1326e-1	—	1.7337e-4	—	7.2194e-8	—	1.6147e-11	—
20	2.4939e-1	2.16	1.1357e-4	4.01	3.8373e-8	5.99	6.9491e-12	8.00
22	2.0127e-1	2.24	7.6831e-5	4.10	2.1537e-8	6.06	3.2241e-12	8.05
24	1.6700e-1	2.14	5.4403e-5	3.96	1.2823e-8	5.96	1.6112e-12	7.97
26	1.4051e-1	2.15	3.9307e-5	4.06	7.9016e-9	6.04	8.4199e-13	8.10
28	1.1966e-1	2.16	2.9199e-5	4.01	5.0721e-9	5.98	4.6896e-13	7.90
30	1.0355e-1	2.09	2.2132e-5	4.01	3.3503e-9	6.01	2.4158e-13	9.61

4. Numerical Experiments

The purpose of this section is to validate the superconvergence results of this paper. We use the DG method and carry out several experiments by numerically solving the model problem (1a) subject to either mixed Dirichlet-Neumann or purely Dirichlet boundary conditions. In addition, we use the DG method to solve a nonlinear boundary-value problem to show numerically that the achieved results are still valid for the nonlinear case. In all computations, we have used uniform and non-uniform meshes and observed similar results. We compute the maximum DG errors $\|e\|_\infty^*$ at the downwind point of each element I_k and then take the maximum over all

TABLE 2. Maximum errors $\|e'\|_\infty^*$ at the upwind points for Example 4.1 using p =1, 2, 3 and 4.

N	$p=1$		$p=2$		$p=3$		$p=4$	
	$\|e'\|_\infty^*$	Order	$\|e'\|_\infty^*$	Order	$\|e'\|_\infty^*$	Order	$\|e'\|_\infty^*$	Order
18	6.7456e-1	—	4.0220e-4	—	1.5828e-7	—	3.8419e-11	—
20	5.3758e-1	2.15	2.6334e-4	4.01	8.4109e-8	6.00	1.6506e-11	8.01
22	4.3449e-1	2.23	1.7829e-4	4.09	4.7222e-8	6.05	7.6525e-12	8.06
24	3.6076e-1	2.13	1.2614e-4	3.98	2.8103e-8	5.96	3.8263e-12	7.97
26	3.0373e-1	2.14	9.1169e-5	4.05	1.7322e-8	6.04	2.0073e-12	8.05
28	2.5883e-1	2.15	6.7711e-5	4.01	1.1117e-8	5.98	1.1156e-12	7.93
30	2.2405e-1	2.09	5.1317e-5	4.01	7.3431e-9	6.01	6.3594e-13	8.14

elements I_k, $k = 1, \ldots, N$. Similarly, the maximum DG errors $\|e'\|_\infty^*$ is computed at upwind point of each element and by taking the maximum over all elements, *i.e.*,

$$\|e\|_\infty^* = \max_{1\le k\le N} |e(x_k^-)|, \quad \|e'\|_\infty^* = \max_{1\le k\le N} |e'(x_{k-1}^+)|.$$

Example 4.1. We consider the following convection-diffusion problem subject to the mixed boundary conditions

$$-0.5u'' + u' = (x-1)\sinh(x) + (1-0.5x)\cosh(x), \quad x \in [0,4],$$
$$u(0) = 0, \quad u'(4) = 4\sinh(4) + \cosh(4).$$

The exact solution is given by $u(x) = x\cosh(x)$. We solve this problem using the DG method on uniform meshes having N = 10, 20, 30, 40, 50, 60 elements and using the spaces P^p with $p = 1, 2, 3,$ and 4. In Figure 1, we plot $\|e\|$ and $\|e'\|$ versus N in a log-log graph in order to obtain the convergence rates for $\|e\|$ and $\|e'\|$. We conclude that $\|e\| = \mathcal{O}(h^{p+1})$ and $\|e'\| = \mathcal{O}(h^p)$. In Tables 1 and 2, we present the maximum errors $\|e\|_\infty^*$ and $\|e'\|_\infty^*$ as well as their order of convergence. These tables show that the DG errors e and e' are $\mathcal{O}(h^{2p})$ superconvergent, respectively, at the downwind and upwind endpoints of each subinterval. These results are in full agreement with the theory.

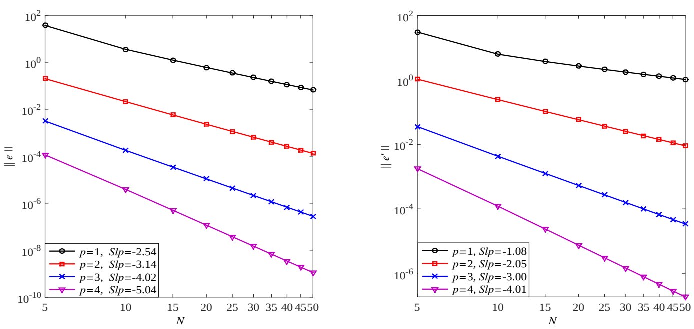

Figure 2: The L^2-norm of the error $\|e\|$ (left) and the derivative of error $\|e'\|$ (right) for Example 4.2 versus N using $p = 1, 2, 3, 4$.

TABLE 3. Maximum errors $\|e\|_\infty^*$ at the downwind points for Example 4.2 using p =1, 2, 3 and 4.

N	$p=1$		$p=2$		$p=3$		$p=4$	
	$\|e\|_\infty^*$	Order	$\|e\|_\infty^*$	Order	$\|e\|_\infty^*$	Order	$\|e\|_\infty^*$	Order
8	3.8100	—	2.1668e-3	—	8.8800e-7	—	1.0869e-9	—
10	1.8548	3.22	8.0881e-4	4.41	2.2702e-7	6.11	1.7250e-10	8.24
12	1.0544	3.09	3.7171e-4	4.26	7.4924e-8	6.08	3.9737e-11	8.05
14	6.5986e-1	3.04	1.9227e-4	4.27	2.9470e-8	6.05	1.1184e-11	8.22
16	4.4142e-1	3.01	1.0940e-4	4.22	1.3177e-8	6.02	3.7677e-12	8.14
18	3.1851e-1	2.77	6.6982e-5	4.16	6.4806e-9	6.02	1.4522e-12	8.09
20	2.3833e-1	2.75	4.3076e-5	4.18	3.4356e-9	6.02	6.0485e-13	8.31

TABLE 4. Maximum errors $\|e'\|_\infty^*$ at the upwind points for Example 4.2 using p =1, 2, 3 and 4.

N	$p=1$		$p=2$		$p=3$		$p=4$	
	$\|e'\|_\infty^*$	Order	$\|e'\|_\infty^*$	Order	$\|e'\|_\infty^*$	Order	$\|e'\|_\infty^*$	Order
8	3.9551	—	2.5591e-3	—	1.4520e-6	—	8.1256e-10	—
10	1.9243	3.22	9.6153e-4	4.38	3.7290e-7	6.09	1.2612e-10	8.34
12	1.0932	3.10	4.4298e-4	4.25	1.2341e-7	6.06	2.8877e-11	8.08
14	6.8375e-1	3.04	2.2987e-4	4.25	4.8608e-8	6.04	8.0131e-12	8.31
16	4.5717e-1	3.01	1.3107e-4	4.20	2.1739e-8	6.02	2.6805e-12	8.20
18	3.2945e-1	2.78	8.0336e-5	4.15	1.0694e-8	6.02	1.0294e-12	8.12
20	2.4623e-1	2.76	5.1748e-5	4.17	5.6711e-9	6.02	4.3965e-13	8.07

Example 4.2. In this example, we consider the following convection-diffusion problem subject to the Dirichlet boundary conditions

$$-u'' + u' = e^x(\sin(x) - \cos(x)), \quad x \in [0, \pi],$$
$$u(0) = u(\pi) = 0.$$

The exact solution is given by $u(x) = e^x \sin x$. We solve this problem using the DG method on uniform meshes having N = 5, 10, 15, 20, 25, 30, 35, 40, 45, 50 elements and using the spaces P^p with $p = 1, 2, 3$ and 4. In order to obtain the convergence rates for e and e', we plot $\|e\|$ and $\|e'\|$ versus N in Figure 2 using a log-log graph. We observe that $\|e\| = \mathcal{O}(h^{p+1})$ and $\|e'\| = \mathcal{O}(h^p)$. The maximum errors $\|e\|_\infty^*$ and $\|e'\|_\infty^*$ as well as their order of convergence shown in Tables 3 and 4 indicate that the DG errors e and e' are $\mathcal{O}(h^{2p})$ superconvergent, respectively, at the downwind and upwind points of each element. These results confirm the theoretical findings of this paper.

Example 4.3. We consider the following singularly perturbed problem subject to the Dirichlet boundary conditions

$$-\epsilon u'' + u' = \exp(x), \quad x \in [0, 1],$$
$$u(0) = 0, \quad u(1) = 0,$$

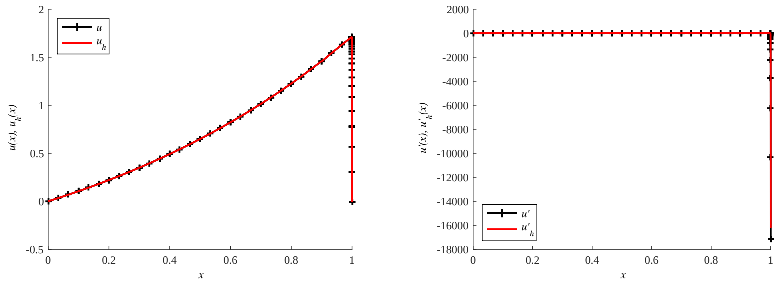

Figure 3: The solutions u and u_h (left) and u' and u'_h (right) for Example 4.3 with $\epsilon = 10^{-4}$ using N=20 with $p = 2$.

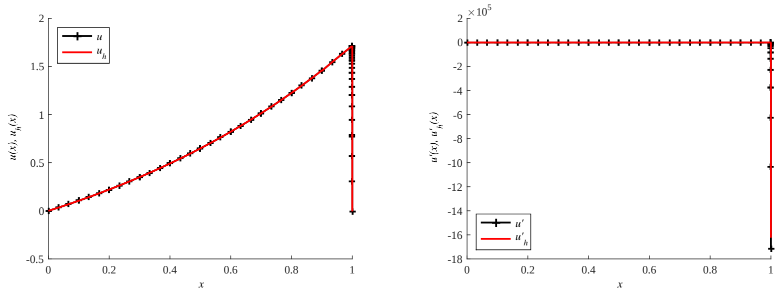

Figure 4: The solutions u and u_h (left) and u' and u'_h (right) for Example 4.3 with $\epsilon = 10^{-6}$ using N=20 with $p = 2$.

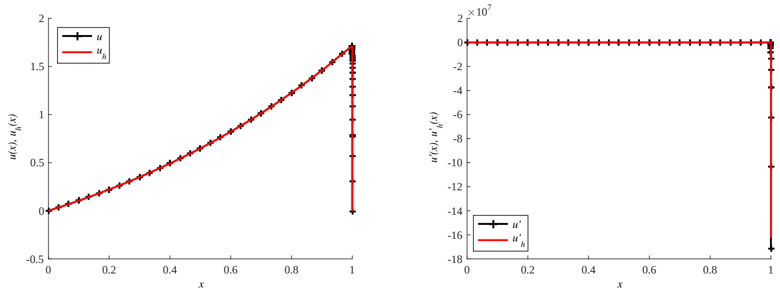

Figure 5: The solutions u and u_h (left) and u' and u'_h (right) for Example 4.3 with $\epsilon = 10^{-8}$ using N=20 with $p = 2$.

where ϵ is a small positive parameter. The exact solution is given by

$$u(x) = \begin{cases} \frac{(1-\exp(1))\exp(\frac{x-1}{\epsilon})+\left(1-\exp(-\frac{1}{\epsilon})\right)\exp(x)+\exp(1-\frac{1}{\epsilon})-1}{(1-\epsilon)\left(1-\exp(-\frac{1}{\epsilon})\right)}, & \epsilon \neq 1, \\ \frac{\exp(1)}{\exp(1)-1}(\exp(x)-1) - x\exp(x), & \epsilon = 1. \end{cases}$$

We note that the true solution has a boundary layer with the width $\mathcal{O}(\epsilon|\ln \epsilon|)$ at the boundary $x = 1$. We solve this problem with $\epsilon = 10^{-4}, 10^{-6}, 10^{-8}$ using the polynomial spaces V_h^p, $p = 1, 2, 3, 4$ on Shishkin meshes [55] having N elements, where N in an even positive integer, and using a mesh transition parameter $\tau = (2p+1)\epsilon \ln(N+1)$ which denotes the approximate width of the boundary layer.

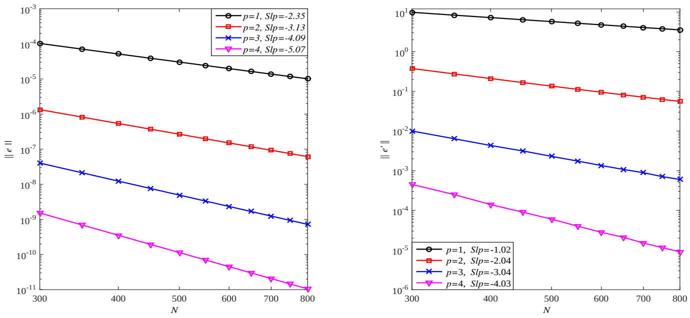

Figure 6: The L^2-norm of the error $\|e\|$ (left) and the derivative of error $\|e'\|$ (right) for Example 4.3 with $\epsilon = 10^{-4}$ versus N using $p = 1, 2, 3, 4$.

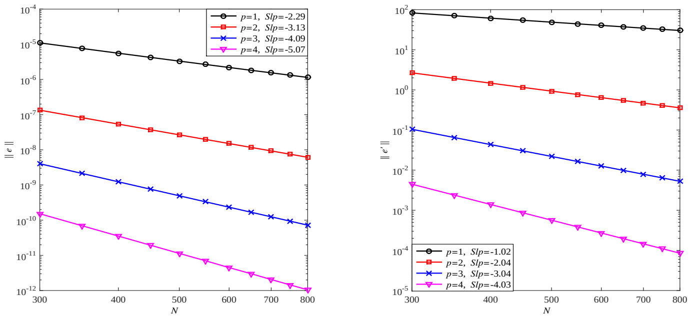

Figure 7: The L^2-norm of the error $\|e\|$ (left) and the derivative of error $\|e'\|$ (right) for Example 4.3 with $\epsilon = 10^{-6}$ versus N using $p = 1, 2, 3, 4$.

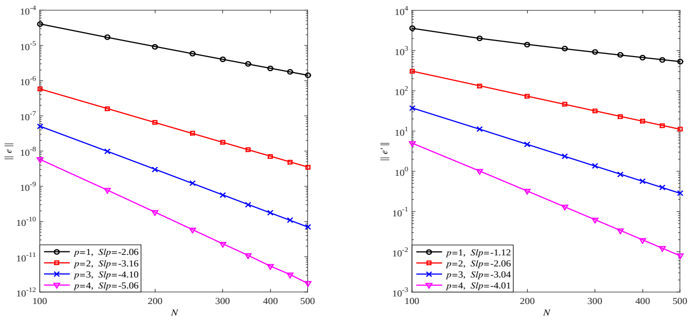

Figure 8: The L^2-norm of the error $\|e\|$ (left) and the derivative of error $\|e'\|$ (right) for Example 4.3 with $\epsilon = 10^{-8}$ versus N using $p = 1, 2, 3, 4$.

The computational domain $[0, 1]$ is divided into two subintervals $[0, 1-\tau]$ and $[\tau, 1]$. Each interval $[0, 1-\tau]$ and $[1-\tau, 1]$ is uniformly subdivided into $\frac{N}{2}$ subintervals which yields a Shishkin mesh.

TABLE 5. Maximum errors $\|e\|_\infty^*$ at the downwind points for Example 4.3 with $\epsilon = 10^{-4}$ using p =1, 2, 3 and 4.

N	$p=1$		$p=2$		$p=3$		$p=4$	
	$\|e\|_\infty^*$	Order	$\|e\|_\infty^*$	Order	$\|e\|_\infty^*$	Order	$\|e\|_\infty^*$	Order
100	1.0198e-1	—	4.4141e-4	—	5.7194e-6	—	9.1705e-8	—
150	2.8468e-2	3.14	6.9654e-5	4.55	4.2989e-7	6.38	3.3129e-9	8.19
200	1.2348e-2	2.90	1.9847e-5	4.36	6.9846e-8	6.31	2.9315e-10	8.42
250	6.8897e-3	2.61	7.6694e-6	4.26	1.7184e-8	6.28	4.7760e-11	8.13
300	4.3709e-3	2.49	3.5507e-6	4.22	5.6021e-9	6.14	1.0874e-11	8.11
350	3.0214e-3	2.39	1.8598e-6	4.19	2.1564e-9	6.19	3.3615e-12	7.61

TABLE 6. Maximum errors $\|e\|_\infty^*$ at the downwind points for Example 4.3 with $\epsilon = 10^{-6}$ using p =1, 2, 3 and 4.

N	$p=1$		$p=2$		$p=3$		$p=4$	
	$\|e\|_\infty^*$	Order	$\|e\|_\infty^*$	Order	$\|e\|_\infty^*$	Order	$\|e\|_\infty^*$	Order
30	1.7958e+0	—	8.2734e-3	—	3.1700e-4	—	1.5560e-5	—
50	1.7400e-1	4.56	8.9969e-4	4.34	1.5304e-5	5.93	3.2374e-7	7.58
70	5.7630e-2	3.28	1.9625e-4	4.52	1.8587e-6	6.26	2.1665e-8	8.03
90	2.6473e-2	3.09	6.2919e-5	4.52	3.6852e-7	6.43	2.6397e-9	8.37
110	1.4746e-2	2.91	2.6267e-5	4.35	1.0328e-7	6.33	4.5072e-10	8.80
130	9.3811e-3	2.70	1.2783e-5	4.31	3.6534e-8	6.22	1.2223e-10	7.81

TABLE 7. Maximum errors $\|e\|_\infty^*$ at the downwind points for Example 4.3 with $\epsilon = 10^{-8}$ using p =1, 2, 3 and 4.

N	$p=1$		$p=2$		$p=3$		$p=4$	
	$\|e\|_\infty^*$	Order	$\|e\|_\infty^*$	Order	$\|e\|_\infty^*$	Order	$\|e\|_\infty^*$	Order
50	5.4372e-1	—	3.2964e-3	—	9.0989e-5	—	3.1662e-6	—
60	2.6121e-1	4.02	1.4867e-3	4.36	3.0564e-5	5.98	7.8384e-7	7.65
70	1.5115e-1	3.54	7.4961e-4	4.44	1.1890e-5	6.12	2.3076e-7	7.93
80	9.6836e-2	3.33	4.1090e-4	4.50	5.1766e-6	6.22	7.6172e-8	8.30
90	6.6296e-2	3.21	2.4041e-4	4.55	2.4563e-6	6.32	2.9924e-8	7.93
100	4.7594e-2	3.14	1.4819e-4	4.59	1.2494e-6	6.41	1.1953e-8	8.70

We plot the DG solution and its derivative in Figures 3-5 using $N = 20$, $p = 2$, and $\epsilon = 10^{-4}, 10^{-6}, 10^{-8}$. We observe that the DG solutions do not have any oscillatory behavior near the boundary layer at the outflow boundary $x = 1$. The L^2 error norms shown in Figures 6-8, respectively, exhibit an $O(h^{p+1})$ and $O(h^p)$ convergence rates. In Tables 5-7 and 8-10, respectively, we present the maximum errors $\|e\|_\infty^*$ and $\|e'\|_\infty^*$ as well as their order of convergence. These tables show that the DG errors e and e' are $\mathcal{O}(h^{2p})$ superconvergent, respectively, at the downwind and upwind endpoints of each subinterval. These results indicate that the analysis techniques in this paper is still valid for singularly perturbed problems. The analysis remains an open problem for the DG method and will be investigated in the future.

TABLE 8. Maximum errors $\|e'\|_\infty^*$ at the upwind points for Example 4.3 with $\epsilon = 10^{-4}$ using p =1, 2, 3 and 4.

N	$p=1$		$p=2$		$p=3$		$p=4$	
	$\|e\|_\infty^*$	Order	$\|e\|_\infty^*$	Order	$\|e\|_\infty^*$	Order	$\|e\|_\infty^*$	Order
100	1.0198e+3	—	4.4142e+0	—	5.7195e-2	—	9.1705e-4	—
150	2.8469e+2	3.14	6.9656e-1	4.55	4.2990e-3	6.38	3.3119e-5	8.19
200	1.2349e+2	2.90	1.9848e-1	4.36	6.9849e-4	6.31	2.9428e-6	8.41
250	6.8902e+1	2.61	7.6703e-2	4.26	1.7185e-4	6.28	4.7761e-7	8.14
300	4.3714e+1	2.49	3.5513e-2	4.22	5.6027e-5	6.14	1.0356e-7	8.38
350	3.0218e+1	2.39	1.8602e-2	4.19	2.1561e-5	6.19	3.1369e-8	7.75

TABLE 9. Maximum errors $\|e'\|_\infty^*$ at the upwind points for Example 4.3 with $\epsilon = 10^{-6}$ using p =1, 2, 3 and 4.

N	$p=1$		$p=2$		$p=3$		$p=4$	
	$\|e\|_\infty^*$	Order	$\|e\|_\infty^*$	Order	$\|e\|_\infty^*$	Order	$\|e\|_\infty^*$	Order
30	1.7958e+6	—	8.2734e+3	—	3.1700e+2	—	1.5560e+1	—
50	1.7400e+5	4.56	8.9969e+2	4.34	1.5304e+1	5.93	3.2374e-1	7.58
70	5.7630e+4	3.28	1.9625e+2	4.52	1.8586e+0	6.26	2.1665e-2	8.03
90	2.6474e+4	3.09	6.2919e+1	4.52	3.6840e-1	6.43	2.7357e-3	8.23
110	1.4746e+4	2.91	2.6267e+1	4.35	1.0328e-1	6.33	4.5072e-4	8.98
130	9.3811e+3	2.70	1.2783e+1	4.31	3.6534e-2	6.22	9.1790e-5	9.52

TABLE 10. Maximum errors $\|e'\|_\infty^*$ at the upwind points for Example 4.3 with $\epsilon = 10^{-8}$ using p =1, 2, 3 and 4.

N	$p=1$		$p=2$		$p=3$		$p=4$	
	$\|e\|_\infty^*$	Order	$\|e\|_\infty^*$	Order	$\|e\|_\infty^*$	Order	$\|e\|_\infty^*$	Order
50	5.4372e+7	—	3.2964e+5	—	9.0989e+3	—	3.1662e+2	—
60	2.6121e+7	4.02	1.4867e+5	4.36	3.0564e+3	5.98	7.8384e+1	7.66
70	1.5115e+7	3.54	7.4962e+4	4.44	1.1890e+3	6.12	2.3076e+1	7.93
80	9.6836e+6	3.33	4.1091e+4	4.50	5.1766e+2	6.22	7.6172e+0	8.30
90	6.6296e+6	3.21	2.4040e+4	4.55	2.4563e+2	6.32	2.9924e+0	7.93
100	4.7594e+6	3.14	1.4819e+4	4.59	1.2494e+2	6.41	1.1953e+0	8.70

Example 4.4. In this final example, we demonstrate numerically that the theoretical results stated in this paper are still valid for the nonlinear case which will be the subject of our future work. We consider the following nonlinear second-order boundary value problem subject to the Dirichlet boundary conditions

$$\begin{aligned} &u'' + \ln(u) = (2 + 4x^2)\exp(x^2) + x^2, \quad x \in [0, 2], \\ &u(0) = 1, \quad u(2) = \exp(4) \end{aligned}$$

The exact solution is given by $u(x) = \exp(x^2)$. We solve this problem using the DG method on uniform meshes having $N = 18, 20, \cdots, 30$ steps and with $p = 1, 2, 3, 4$. The errors $||e||$ and $\|e'\|$ versus N shown in Figure 9, respectively, exhibit an $O(h^{p+1})$ and $O(h^p)$ convergence rates. The maximum errors $\|e\|_\infty^*$ and $\|e'\|_\infty^*$ as well as their order of convergence presented in Tables 11 and 12 show that the DG errors e and e' are $\mathcal{O}(h^{2p})$ superconvergent, respectively, at the downwind and upwind endpoints of every element. These results are in full agreement with the theory.

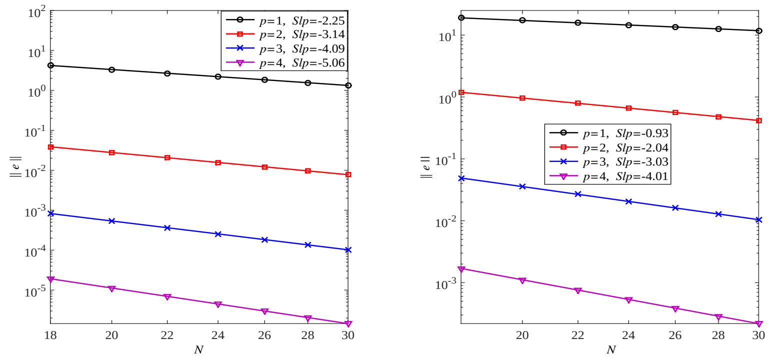

Figure 9: The L^2-norm of the error $\|e\|$ (left) and the derivative of error $\|e'\|$ (right) for Example 4.4 versus N using $p = 1, 2, 3, 4$.

TABLE 11. Maximum errors $\|e\|_\infty^*$ at the downwind points for Example 4.4 using p =1, 2, 3 and 4.

N	$p=1$		$p=2$		$p=3$		$p=4$	
	$\|e\|_\infty^*$	Order	$\|e\|_\infty^*$	Order	$\|e\|_\infty^*$	Order	$\|e\|_\infty^*$	Order
18	1.5450e-2	—	3.6558e-5	—	4.5515e-8	—	4.0144e-11	—
20	1.2698e-2	1.86	2.3395e-5	4.23	2.3992e-8	6.07	1.6956e-11	8.17
22	1.0601e-2	1.89	1.5654e-5	4.21	1.3491e-8	6.04	7.8355e-12	8.09
24	8.9779e-3	1.91	1.0865e-5	4.19	7.9667e-9	6.05	3.7836e-12	8.36
26	7.7040e-3	1.91	7.7743e-6	4.18	4.9004e-9	6.07	2.0002e-12	7.96
28	6.6777e-3	1.93	5.7154e-6	4.15	3.1309e-9	6.04	1.0791e-12	8.32
30	5.8402e-3	1.94	4.2984e-6	4.12	2.0626e-9	6.04	6.2084e-13	8.01

TABLE 12. Maximum errors $\|e'\|_\infty^*$ at the upwind points for Example 4.4 using p =1, 2, 3 and 4.

N	$p=1$		$p=2$		$p=3$		$p=4$	
	$\|e'\|_\infty^*$	Order	$\|e'\|_\infty^*$	Order	$\|e'\|_\infty^*$	Order	$\|e'\|_\infty^*$	Order
18	3.7768e-2	—	4.4035e-5	—	4.8477e-8	—	3.7273e-11	—
20	3.1007e-2	1.87	2.8317e-5	4.19	2.5662e-8	6.03	1.5829e-11	8.12
22	2.5914e-2	1.88	1.9039e-5	4.16	1.4430e-8	6.03	7.3066e-12	8.11
24	2.1982e-2	1.89	1.3275e-5	4.14	8.5308e-9	6.04	3.5447e-12	8.31
26	1.8884e-2	1.90	9.5393e-6	4.12	5.2597e-9	6.04	1.8767e-12	7.94
28	1.6399e-2	1.90	7.0322e-6	4.11	3.3614e-9	6.04	1.0138e-12	8.30
30	1.4375e-2	1.91	5.4382e-6	3.72	2.2157e-9	6.04	5.8661e-13	7.93

5. Concluding remarks

In this paper we studied the convergence and superconvergence properties of a DG method for one-dimensional convection-diffusion problems. We proved that the DG solution and its derivative exhibit optimal $\mathcal{O}(h^{p+1})$ and $\mathcal{O}(h^p)$ convergence rates in the L^2-norm, respectively, when p-degree piecewise polynomials with $p \geq 1$ are used. We further proved that the p-degree DG solution and its derivative are $\mathcal{O}(h^{2p})$ superconvergent at the downwind and upwind points, respectively. Numerical experiments demonstrate that the theoretical rates are optimal and our results hold for some nonlinear problems. We are currently investigating the superconvergence properties of the DG method applied to higher-order boundary-value problems. We plan to study the superconvergence properties and the asymptotic exactness of *a posteriori* error estimates for DG methods applied to nonlinear problems and to two-dimensional convection-diffusion problems on rectangular and triangular meshes. Extending the error analysis to problems on tetrahedral meshes will be investigated in the future.

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Department of Mathematics, University of Nebraska at Omaha, Omaha, NE 68182, USA

E-mail: `mbaccouch@unomaha.edu`

URL: `http://www.unomaha.edu/baccouch/`

Department of Mathematics & Natural Sciences, Gulf University for Science & Technology, P.O. Box 7207, Hawally 32093, Kuwait